Handbook of Mathematical Psychology
edited by
Luce, Bush and Galanter
Published by John Wiley & Sons, Inc., 1963

10

Stimulus Sampling Theory[1]

Richard C. Atkinson and
William K. Estes
Stanford University

1. *Preparation of this chapter was supported in part by Contract Nonr-908(16) between the Office of Naval Research and Indiana University, Grant M-5184 from the National Institute of Mental Health to Stanford University, and Contract Nonr-225(17) between the Office of Naval Research and Stanford University.*

Contents

Stimulus Sampling Theory

Stimulus sampling theory is concerned with providing a mathematical language in which we can state assumptions about learning and performance in relation to stimulus variables. A special advantage of the formulations to be discussed is that their mathematical properties permit application of the simple and elegant theory of Markov chains (Feller, 1957; Kemeny, Snell, & Thompson, 1957; Kemeny & Snell, 1959) to the tasks of deriving theorems and generating statistical tests of the agreement between assumptions and data. This branch of learning theory has developed in close interaction with certain types of experimental analyses; consequently it is both natural and convenient to organize this presentation around the theoretical treatments of a few standard reference experiments.

At the level of experimental interpretation most contemporary learning theories utilize a common conceptualization of the learning situation in terms of *stimulus*, *response*, and *reinforcement*. The stimulus term of this triumvirate refers to the environmental situation with respect to which behavior is being observed, the response term to the class of observable behaviors whose measurable properties change in some orderly fashion during learning, and the reinforcement term to the experimental operations or events believed to be critical in producing learning. Thus, in a simple paired-associate experiment concerned with the learning of English equivalents to Russian words, the stimulus might consist in presentation of the printed Russian word alone, the response measure in the relative frequency with which the learner is able to supply the English equivalent from memory, and reinforcement in paired presentation of the stimulus and response words.

In other chapters of this *Handbook*, and in the general literature on learning theory, the reader will encounter the notions of sets of responses and sets of reinforcing events. In the present chapter mathematical sets are used to represent certain aspects of the stimulus situation. It should be emphasized from the outset, however, that the mathematical models to be considered are somewhat abstract and that the empirical interpretations of stimulus sets and their elements are not to be considered fixed and immutable. Two main types of interpretations are discussed: in one the empirical correspondent of a stimulus element is the full pattern of stimulation effective on a given trial; in the other the correspondent of an element is a component, or aspect, of the full pattern of stimulation. In the first, we speak of "pattern models" and in the second, of "component models" (Estes, 1959b).

There are a number of ways in which characteristics of the stimulus situation are known to affect learning and transfer. Rates and limits of conditioning and learning generally depend on stimulus magnitude, or intensity, and on stimulus variability from trial to trial. Retention and transfer of learning depend on the similarity, or communality, between the stimulus situations obtaining during training and during the test for retention or transfer. These aspects of the stimulus situation can be given direct and natural representations in terms of mathematical sets and relations between sets.

The basic notion common to all stimulus sampling theories is the conceptualization of the totality of stimulus conditions that may be effective during the course of an experiment in terms of a mathematical set. Although it is not a necessary restriction, it is convenient for mathematical reasons to deal only with finite sets, and this limitation is assumed throughout our presentation. Stimulus variability is taken into account by assuming that of the total population of stimuli available in an experimental situation generally only a part actually affects the subject on any one trial. Translating this idea into the terms of a stimulus sampling model, we may represent the total population by a set of "stimulus elements" and the stimulation effective on any one trial by a sample from this set. Many of the simple mathematical properties of the models to be discussed arise from the assumption that these trial samples are drawn randomly from the population, with all samples of a given size having equal probabilities. Although it is sometimes convenient and suggestive to speak in such terms, we should not assume that the stimulus elements are to be identified with any simple neurophysiological unit, as, for example, receptor cells. At the present stage of theory construction we mean to assume only that certain properties of the set-theoretical model represent certain properties of the process of stimulation. If these assumptions prove to be sufficiently well substantiated when the model is tested against behavioral data, then it will be in order to look for neurophysiological variables that might underlie the correspondences. Just as the ratio of sample size to population size is a natural way of representing stimulus variability, sample size per se may be taken as a correspondent of stimulus intensity, and the amount of overlap (i.e., proportion of common elements) between two stimulus sets may be taken to represent the degree of communality between two stimulus situations.

Our concern in this chapter is not to survey the rapidly developing area of stimulus sampling theory but simply to present some of the fundamental mathematical techniques and illustrate their applications. For general background the reader is referred to Bush (1960), Bush & Estes (1959), Estes (1959a, 1962), and Suppes & Atkinson (1960). We shall consider first, and in some detail, the simplest of all learning models—the pattern model for simple learning. In this model the population of available stimuli is assumed to comprise a set of distinct stimulus patterns, exactly one of which is sampled on each trial. In the important special case of the one-element model it is assumed that there is only one such pattern and that it recurs intact at the beginning of each experimental trial. Granting that the one-element model represents a radical idealization of even the most simplified conditioning situations, we shall find that it is worthy of study not only for expositional purposes but also for its value as an analytic device in relation to certain types of learning data. After a relatively thorough treatment of pattern models for simple acquisition and for learning under probabilistic reinforcement schedules, we shall take up more briefly the conceptualization of generalization and transfer; component models in which the patterns of stimulation effective on individual trials are treated not as distinct elements but as overlapping samples from a common population; and, finally, some examples of the more complex multiple-process models that are becoming increasingly important in the analysis of discrimination learning, concept formation, and related phenomena.

1. ONE-ELEMENT MODELS

We begin by considering some one-element models that are special cases of the more general theory. These examples are especially simple mathematically and provide us with the opportunity to develop some mathematical tools that will be necessary in later discussions. Application of these models is appropriate when the stimulus situation is sufficiently stable from trial to trial that it may be theoretically represented (to a good approximation) by a single stimulus element which is sampled with probability 1 on each trial. At the start of a trial the element is in one of several possible conditioning states; it may or may not remain in this conditioning state, depending on the reinforcing event for that trial. In the first part of this section we consider a model for paired-associate learning. In the second part we consider a model for a two-choice learning situation involving a probabilistic reinforcement schedule. The models generate some predictions that are undoubtedly incorrect, except possibly under ideal experimental conditions; nevertheless, they provide a useful introduction to more general cases which we pursue in Section 2.

1.1 Learning of a Single Stimulus-Response Association

Imagine the simplest possible learning situation. A single stimulus pattern, S, is to be presented on each of a series of trials and each trial is to terminate with reinforcement of some designated response, the "correct response" in this situation. According to stimulus sampling theory, learning occurs in an all-or-none fashion with respect to S.

1. If the correct response is not originally conditioned to ("connected to") S, then, until learning occurs, the probability of the correct response is zero.
2. There is a fixed probability c that the reinforced response will become conditioned to S on any trial.
3. Once conditioned to S, the correct response occurs with probability 1 on every subsequent trial.

These assumptions constitute the simplest case of the "one-element pattern model." Learning situations that completely meet the specifications laid down above are as unlikely to be realized in psychological experiments as perfect vacuums or frictionless planes in the physics laboratory. However, reasonable approximations to these conditions can be attained. The requirement that the same stimulus pattern be reproduced on each trial is probably fairly well met in the standard paired-associate experiment with human subjects. In one such experiment, conducted in the laboratory of one of the writers (W. K. E.), the stimulus member of each item was a trigram and the correct response an English word, for example,

S	R .
xvk	house

On a reinforced trial the stimulus and response members were exposed together, as shown. Then, after several such items had received a single reinforcement, each of the stimuli was presented alone, the subject being instructed to give the correct response from memory, if he could. Then each item was given a second reinforcement, followed by a second test, and so on.

According to the assumptions of the one-element pattern model, a subject should be expected to make an incorrect response on each test with a given stimulus until learning occurs, then a correct response on every subsequent trial; if we represent an error by a 1 and a correct response by a 0, the protocol for an individual item over a series of trials should, then, consist in a sequence of 0's preceded in most cases by a sequence of 1's. Actual protocols for several subjects are shown below:

a	0	0	0	0	0	0	0	0	0	0
b	1	1	1	1	1	1	1	1	1	1
c	1	0	0	0	0	0	0	0	0	0
d	0	0	0	0	0	0	0	0	0	0
e	1	1	0	0	0	0	0	0	0	0
f	1	1	0	0	0	0	0	0	0	0
g	1	1	1	1	1	0	0	0	0	0
h	1	0	0	0	0	0	1	0	0	0
i	1	1	1	1	0	1	1	0	0	0

The first seven of these correspond perfectly to the idealized theoretical picture; the last two deviate slightly. The proportion of "fits" and "misfits" in this sample is about the same as in the full set of 80 cases from which the sample was taken. The occasional lapses, that is, errors following correct responses, may be symptomatic of a forgetting process that should be incorporated into the theory, or they may be simply the result of minor uncontrolled variables in the experimental situation which are best ignored for theoretical purposes. Without judging this issue, we may conclude that the simple one-element model at least merits further study.

Before we can make quantitative predictions we need to know the value of the conditioning parameter c. Statistical learning theory includes no formal axioms that specify precisely what variables determine the value of c, but on the basis of considerable experience we can safely assume that this parameter will vary with characteristics of the populations of subjects and items represented in a particular experiment. An estimate of the value of c for the experiment under consideration is easy to come by. In the full set of 80 cases (40 subjects, each tested on two items) the proportion of correct responses on the test given after a single reinforcement was 0.39. According to the model, the probability is c that a reinforced response will become conditioned to its paired stimulus; consequently c is the expected proportion of successful conditionings out of 80 cases, and therefore the expected proportion of correct responses on the subsequent test. Thus we may simply take the observed proportion 0.39 as an estimate of c.

In order to test the model, we need now to derive theoretical expressions for other aspects of the data. Suppose we consider the sequences of correct and incorrect responses, 000, 001, etc., on the first three trials. According to the model, a correct response should never be followed by an error, so the probability of the sequence 000 is simply c, and the probabilities of 001, 010, 011, and 101 are all zero. To obtain an error on the first trial followed by a correct response on the second, conditioning must fail on the first reinforcement but occur on the second, and this joint event has probability $(1 - c)c$. Similarly, the probability that the first correct response will occur on the third trial is given by $(1 - c)^2c$ and the probability of no correct response in three trials by $(1 - c)^3$. Substituting the estimate 0.39 for c in each of these expressions, we obtain the predicted

Table 1 Observed and Predicted (One-Element Model) Values for Response Sequences Over First Three Trials of a Paired-Associate Experiment

Sequence*	Observed Proportions	Theoretical Proportions
000	0.36	0.39
001	0.02	0
010	0.01	0
011	0	0
100	0.27	0.24
101	0	0
110	0.11	0.14
111	0.23	0.23

* 0 = correct response
1 = error

values which are compared with the corresponding empirical values for this experiment in Table 1. The correspondences are seen to be about as close as could be expected with proportions based on 80 response sequences.

1.2 Paired-Associate Learning

In order to apply the one-element model to paired-associate experiments involving fixed lists of items, it is necessary to adjust the "boundary conditions" appropriately. Consider, for example, an experiment reported by Estes, Hopkins, and Crothers (1960). The task assigned their subjects was to learn associations between the numbers 1 through 8, serving as responses, and eight consonant trigrams, serving as stimuli. Each subject was given two practice trials and two test trials. On the first practice trial the eight syllable-number pairs were exhibited singly in a random order. Then a test was given, the syllables alone being presented singly in a new random order and the subjects attempting to respond to each syllable with the correct number. Four of the syllable-number pairs were presented on a second practice trial, and all eight syllables were included in a final test trial.

In writing an expression for the probability of a correct response on the first test in this experiment, we must take account of the fact that, after the first practice trial, the subjects knew that the responses were the numbers 1 to 8 and were in a position to guess at the correct answers when shown syllables that they had not yet learned. The minimum probability of achieving a correct response to an unlearned item by guessing would be $\frac{1}{8}$. Thus we would have for p_0, the probability of a correct response on the first test,

$$p_0 = c + \frac{1 - c}{8},$$

that is, the probability c that the correct association was formed plus the probability $(1 - c)/8$ that the association was not formed but the correct response was achieved by guessing. Setting this expression equal to the observed proportion of correct responses on the first trial for the twice reinforced items, we readily obtain an estimate of c for these experimental conditions,

$$0.404 = c + (1 - c)(0.125),$$

and so

$$\hat{c} = 0.32.$$

Now we can proceed to derive expressions for the joint probabilities of various combinations of correct and incorrect responses on the first and second tests for the twice reinforced items. For the probability of correct responses to a given item in both tests, we have

$$p_{00} = c + (1 - c)(0.125)c + (1 - c)^2(0.125)^2.$$

With probability c, conditioning occurs on the first reinforced trial, and then correct responses necessarily occur on both tests; with probability $(1 - c)c(0.125)$, conditioning does not occur on the first reinforced trial but does on the second, and a correct response is achieved by guessing on the first test; with probability $(1 - c)^2(0.125)^2$, conditioning occurs on neither reinforced trial but correct responses are achieved by guessing on both tests. Similarly, we obtain

$$p_{01} = (1 - c)^2(0.875)(0.125)$$

$$p_{10} = (1 - c)(0.875)[c + (1 - c)(0.125)]$$

and

$$p_{11} = (1 - c)^2(0.875)^2.$$

Substituting for c in these expressions the estimate computed above, we arrive at the predicted values which we compare with the corresponding observed values below.

	Observed	Predicted
p_{00}	0.35	0.35
p_{01}	0.05	0.05
p_{10}	0.27	0.24
p_{11}	0.33	0.35

Although this comparison reveals some disparities, which we might hope to reduce with a more elaborate theory, it is surprising, to the writers at least, that the patterns of observed response proportions in both experiments considered can be predicted as well as they are by such an extremely simple model.

Ordinarily, experiments concerned with paired-associate learning are not limited to a couple of trials, like those just considered, but continue until the subjects meet some criterion of learning. Under these circumstances it is impractical to derive theoretical expressions for all possible sequences of correct and incorrect responses. A reasonable goal, instead, is to derive expressions for various statistics that can be conveniently computed for the data of the standard experiment; examples of such statistics are the mean and variance of errors per item, frequencies of runs of errors or correct responses, and serial correlation of errors over trials with any given lag. Bower (1961, 1962) carried out the first major analysis of this type for the one-element model. We shall use some of his results to illustrate application of the model to a full "learning-to-criterion" experiment. Essential details of his experiment are as follows: a list of 10 items was learned by 29 undergraduates to a criterion of two consecutive errorless trials. The stimuli were different pairs of consonant letters and the responses were the integers 1 and 2; each response was assigned as correct to a randomly selected five items for each subject. A response was obtained from the subject on each presentation of an item, and he was informed of the correct answer following his response.

As in the preceding application, we shall assume that each item in the list is to be represented theoretically by exactly one stimulus element, which is sampled with probability 1 when the item is presented, and that the correct response to that item is conditioned in an all-or-none fashion. On trial n of the experiment an element is in one of two "conditioning states": In state C the element is conditioned to the correct response; in state $\bar{C}$ the element is not conditioned.

The response the subject makes depends on his conditioning state.

When the element is in state C, the correct response occurs with probability 1. The probability of the correct response when the element is in state $\bar{C}$ depends on the experimental procedure. In Bower's experiment the subjects were told the r responses available to them and each occurred equally often as the to-be-learned response. Therefore we may assume that in the unconditioned state the probability of a correct response is $1/r$, where r is the number of alternative responses.

The conditioning assumptions can readily be restated in terms of the conditioning states:

1. On any reinforced trial, if the sampled element is in state $\bar{C}$, it has probability c of going into state C.
2. The parameter c is fixed in value in a given experiment.
3. Transitions from state C to state $\bar{C}$ have probability zero.

We shall now derive some predictions from the model and compare these with observed data. The data of particular interest will be a subject's sequence of correct and incorrect responses to a specific stimulus item over trials. Similarly, in deriving results from the model we shall consider only an isolated stimulus item and its related sequence of responses. However, when we apply the model to data, we assume that all items in the list are comparable, that is, all items have the same conditioning parameter c and all items start out in the same conditioning state ($\bar{C}$). Consequently the response sequence associated with any given item is viewed as a sample of size 1 from a population of sequences all generated by the same underlying process.

A feature of this model which makes it especially tractable for purposes of deriving various statistics is the fact that the sequences of transitions between states C and $\bar{C}$ constitute a Markov chain. This means that, given the state on any one trial, we can specify the probability of each state on the next trial without regard to the previous history. If we represent by C_n and $\bar{C}_n$ the events that an item is in the conditioned or unconditioned state, respectively, on trial n, and by q_{11} and q_{21} the probabilities of transitions from state C to state C and from $\bar{C}$ to C, respectively, the conditioning assumptions lead directly to the relations[2]

$$q_{11} = \Pr(C_{n+1} \mid C_n) = 1,$$
$$q_{21} = \Pr(C_{n+1} \mid \bar{C}_n) = c,$$

[2] See Feller (1957) for a discussion of conditional probabilities. In brief, if $H_1, \ldots, H_n$ are a set of mutually exclusive events of which one necessarily occurs, then any event A can occur only in conjunction with some H_j. Since the AH_j are mutually exclusive, their probabilities add. Applying the well-known theorem on compound probabilities, we obtain $\Pr(A) = \sum_j \Pr(AH_j) = \sum_j \Pr(A \mid H_j) \Pr(H_j)$.

and

$$Q = \begin{bmatrix} 1 & 0 \\ c & 1 - c \end{bmatrix},$$

where Q is the matrix of one-step transition probabilities, the first row and column referring to C and the second row and column to $\bar{C}$. Now the matrix of probabilities for transitions between any two states in n trials is simply the nth power of Q, as may be verified by mathematical induction (see, e.g., Kemeny, Snell, & Thompson, 1957, p. 327),

$$Q^n = \begin{bmatrix} 1 & 0 \\ 1 - (1 - c)^n & (1 - c)^n \end{bmatrix}.$$

Henceforth we shall assume that all stimulus elements are in state $\bar{C}$ at the onset of the first trial of our experiment. Given that the state is $\bar{C}$ on trial 1, the probability of being in state $\bar{C}$ at the start of trial n is $(1 - c)^{n-1}$, which goes to 0 as n becomes large, for $c > 0$. Thus with probability 1 the subject is eventually to be found in the conditioned state.

Next we prove some theorems about the observable sequence of correct and incorrect responses in terms of the underlying sequence of unobservable conditioning states. We define the response random variable

$$\mathbf{A}_n = \begin{cases} 0 & \text{if a correct response occurred on trial } n, \\ 1 & \text{if an error occurred on trial } n. \end{cases}$$

By our assumed response rule the probabilities of an error, given that the subject is in the conditioned or unconditioned state, respectively, are

$$\Pr(\mathbf{A}_n = 1 \mid C_n) = 0$$

and

$$\Pr(\mathbf{A}_n = 1 \mid \bar{C}_n) = 1 - \frac{1}{r}.$$

To obtain the probability of an error on trial n, namely $\Pr(\mathbf{A}_n = 1)$, we sum these conditional probabilities weighted by the probabilities of being in the respective states:

$$\begin{aligned} \Pr(\mathbf{A}_n = 1) &= \Pr(\mathbf{A}_n = 1 \mid C_n)\Pr(C_n) + \Pr(\mathbf{A}_n = 1 \mid \bar{C}_n)\Pr(\bar{C}_n) \\ &= \left(1 - \frac{1}{r}\right)(1 - c)^{n-1}. \end{aligned} \tag{1}$$

Consider next the infinite sum of the random variables $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots$ which we denote $\bar{\mathbf{A}}$; specifically,

$$\bar{\mathbf{A}} = \sum_{n=1}^{\infty} \mathbf{A}_n.$$

But

$$\begin{aligned} E(\bar{\mathbf{A}}) &= \sum E(\mathbf{A}_n) \\ &= \sum \Pr(\mathbf{A}_n = 1) \\ &= \sum_{n=1}^{\infty} \left(1 - \frac{1}{r}\right)(1 - c)^{n-1} \\ &= \frac{1 - (1/r)}{c} . \end{aligned} \tag{2}$$

Thus the number of errors expected during the learning of any given item is given by Eq. 2.

Equation 2 provides an easy method for estimating c. For any given subject we can obtain his average number of errors over stimulus items, equate this number to the right-hand side of Eq. 2 with $r = 2$, and solve for c. We thereby obtain an estimate of c for each subject, and intersubject differences in learning are reflected in the variability of these estimates. Bower, in analyzing his data, chose to assume that c was the same for all subjects; thus he set $E(\bar{\mathbf{A}})$ equal to the observed number of errors averaged over both list items and subjects and obtained a single estimate of c. This group estimate of c simplifies the computations involved in generating predictions. However, it has the disadvantage that a discrepancy between observed and predicted values may arise as a consequence of assuming equal c's when, in fact, the theory is correct but c varies from subject to subject. Fortunately, Bower has obtained excellent agreement between theory and observation using the group estimate of c and, for the particular conditions he investigated, any increase in precision that might be achieved by individual estimates of c does not seem crucial.

For the experiment described above, Bower reports 1.45 errors per stimulus item averaged over all subjects. Equating $E(\bar{\mathbf{A}})$ in Eq. 2 to 1.45, with $r = 2$, we obtain the estimate $c = 0.344$. All predictions that we derive from the model for this experiment will be based on this single estimate of c. It should be remarked that the estimate of c in terms of Eq. 2 represents only one of many methods that could have been used. The method one selects depends on the properties of the particular estimator (e.g., whether the estimator is unbiased and efficient in relation to other estimators). Parameter estimation is a theory in its own right, and we shall not be able to discuss the many problems involved in the estimation of learning parameters. The reader is referred to Suppes & Atkinson (1960) for a discussion of various methods and their properties. Associated with this topic is the problem of assessing the statistical agreement between data and theory(i. e., the goodness-of-fit between predicted and observed values) once parameters have been estimated. In our analysis of data in this chapter we offer no statistical evaluation of the predictions but simply display the results for the reader's inspection. Our reason is that we present the data only to illustrate features of the theory and its application; these results are not intended to provide a test of the model. However, in rigorous analyses of such models the problem of goodness-of-fit is extremely important and needs careful consideration. Here again the reader is referred to Suppes & Atkinson (1960) for a discussion of some of the problems and possible statistical tests.

By using Eq. 1 with the estimate of c obtained above we have generated the predicted learning curve presented in Fig. 1. The fit is sufficiently close that most of the predicted and observed points cannot be distinguished on the scale of the graph.

As a basis for the derivation of other statistics of total errors, we require an expression for the probability distribution of $\bar{\mathbf{A}}$. To obtain this, we note first that the probability of no errors at all occurring during learning is given by

$$c\left(\frac{1}{r}\right) + (1 - c)\left(\frac{1}{r}\right)^2 c + \ldots = \frac{c}{r}\sum_{i=0}^{\infty}\left(\frac{1-c}{r}\right)^i = \frac{c}{r[1-(1-c)/r]} = \frac{b}{r},$$

where $b = c/[1 - (1 - c)/r]$. This event may arise if a correct response occurs by guessing on the first trial and conditioning occurs on the first reinforcement, if a correct response occurs by guessing on the first two

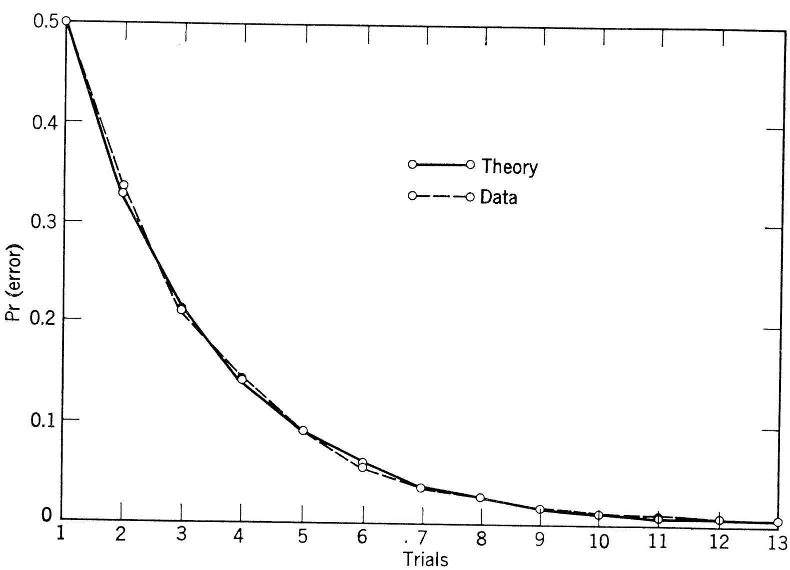

Fig. 1. The average probability of an error on trial n in Bower's paired-associate experiment.

trials and conditioning occurs on the second reinforcement, and so on. Similarly, the probability of no additional errors following an error on any given trial is given by

$$c + c\frac{1-c}{r} + \ldots = c\sum_{i=0}^{\infty}\left(\frac{1-c}{r}\right)^i = \frac{c}{1-(1-c)/r} = b.$$

To have exactly k errors, we must have a first error (if $k > 0$), which has probability $1 - b/r$, $k - 1$ additional errors, each of which has probability $1 - b$, and then no more errors. Therefore the required probability distribution is

$$\Pr(\bar{\mathbf{A}} = 0) = \frac{b}{r} \tag{3}$$

$$\Pr(\bar{\mathbf{A}} = k) = b(1 - b/r)(1 - b)^{k-1}, \qquad \text{for } k \geqslant 1.$$

Equation 3 can be applied to data directly to predict the form of the frequency distribution of total errors. It may also be utilized in deriving, for example, the variance of this distribution. Preliminary to computing the variance, we need the expectation of $\bar{\mathbf{A}}^2$,

$$\begin{aligned} E(\bar{\mathbf{A}}^2) &= \sum_{k=0}^{\infty} k^2 b\left(\frac{r-b}{r}\right)(1-b)^{k-1} \\ &= b\left(\frac{r-b}{r}\right)\sum_{k=0}^{\infty}[k(k-1)+k](1-b)^{k-1} \\ &= (1-b)b\left(\frac{r-b}{r}\right)\sum_{k=0}^{\infty}[k(k-1)+k](1-b)^{k-2}, \end{aligned}$$

where the second step is taken in order to facilitate the summation. Using the familiar expression

$$\sum_{k=0}^{\infty}(1-b)^k = \frac{1}{b}$$

for the sum of a geometric series, together with the relations

$$\frac{d}{db}(1-b)^k = -k(1-b)^{k-1},$$

$$\frac{d^2}{db^2}(1-b)^k = k(k-1)(1-b)^{k-2},$$

and

$$-\sum_{k=0}^{\infty}\frac{d}{db}(1-b)^k = -\frac{d}{db}\sum_{k=0}^{\infty}(1-b)^k = -\frac{d}{db}\left(\frac{1}{b}\right) = \frac{1}{b^2},$$

$$\sum_{k=0}^{\infty}\frac{d^2}{db^2}(1-b)^k = \frac{d^2}{db^2}\sum_{k=0}^{\infty}(1-b)^k = \frac{d^2}{db^2}\left(\frac{1}{b}\right) = \frac{2}{b^3},$$

we obtain

$$E(\bar{\mathbf{A}}^2) = b\left(\frac{r-b}{r}\right)\left[\frac{2(1-b)}{b^3} + \frac{1}{b^2}\right]$$

and

$$\begin{aligned}
\operatorname{Var}(\bar{\mathbf{A}}) &= E(\bar{\mathbf{A}}^2) - [E(\bar{\mathbf{A}})]^2 \\
&= b\left(\frac{r-b}{r}\right)\left[\frac{2(1-b)}{b^3} + \frac{1}{b^2}\right] - \frac{[1-(1/r)]^2}{c^2} \\
&= \left(1 - \frac{1}{r}\right)\frac{(2c - cr + r - 1)}{c^2 r} \\
&= \frac{(r-1)}{rc}\,\frac{(2c - cr + r - 1)}{rc} \\
&= \frac{(r-1)}{rc}\,\frac{(cr + 2c - 2cr + r - 1)}{rc} \\
&= \frac{(r-1)}{rc}\left[1 + \frac{(2c-1)(1-r)}{rc}\right] \\
&= E(\bar{\mathbf{A}})[1 + E(\bar{\mathbf{A}})(1 - 2c)]. \qquad (4)
\end{aligned}$$

Inserting in Eq. 4 the estimates $E(\bar{\mathbf{A}}) = 1.45$ and $c = 0.344$ from Bower's data, we obtain 1.44 for the predicted standard deviation of total errors, which may be compared with the observed value of 1.37.

Another useful statistic of the error sequence is $E(\mathbf{A}_n\mathbf{A}_{n+k})$; namely, the expectation of the product of error random variables on trials n and $n + k$. This quantity is related to the autocorrelation between errors on trials $n + k$ and trial n. By elementary probability theory,

$$\begin{aligned}
E(\mathbf{A}_n\mathbf{A}_{n+k}) &= E(\mathbf{A}_{n+k} \mid \mathbf{A}_n)E(\mathbf{A}_n) \\
&= \Pr(\mathbf{A}_{n+k} = 1 \mid \mathbf{A}_n = 1)\Pr(\mathbf{A}_n = 1).
\end{aligned}$$

But for an error to occur on trial $n + k$ conditioning must have failed to occur during the intervening k trials and the subject must have guessed incorrectly on trial $n + k$. Hence

$$\Pr(\mathbf{A}_{n+k} = 1 \mid \mathbf{A}_n = 1) = (1-c)^k\left(1 - \frac{1}{r}\right).$$

Substitution of this result into the preceding expression, along with the result presented in Eq. 1, yields the following expression:

$$\begin{aligned}
E(\mathbf{A}_n\mathbf{A}_{n+k}) &= \left(1 - \frac{1}{r}\right)(1-c)^k(1-c)^{n-1}\left(1 - \frac{1}{r}\right) \\
&= \left(1 - \frac{1}{r}\right)^2(1-c)^{n+k-1}. \qquad (5)
\end{aligned}$$

A convenient statistic for comparison with data (directly related to the average autocorrelation of errors with lag k, but easier to compute) is obtained by summing the cross product of $\mathbf{A}_n$ and $\mathbf{A}_{n+k}$ over all trials. We define c_k as the mean of this random variable, where

$$\begin{aligned} c_k &= \sum_{n=1}^{\infty} E(\mathbf{A}_{n+k}\mathbf{A}_n) \\ &= E(\bar{\mathbf{A}})\left(1 - \frac{1}{r}\right)(1 - c)^k. \end{aligned} \tag{6}$$

To be explicit, consider the following response protocol running in time from left to right: 1101010010000. The observed values for c_k are $c_1 = 1$, $c_2 = 2$, $c_3 = 2$, and so on.

The predictions for c_1, c_2, and c_3 computed from the c estimate given above were 0.479, 0.310, and 0.201. Bower's observed values were 0.486, 0.292, and 0.187.

Next we consider the distribution of the number of errors between the kth and $(k + 1)$st success. The methods to be used in deriving this result are general and can be used to derive the distribution of errors between the kth and $(k + m)$th success for any nonnegative integer m. The only limitation is that the expressions become unwieldy as m increases. We shall define $\mathbf{J}_k$ as the random variable for the number of errors between the kth and $(k + 1)$st success; its values are 0, 1, 2, ... An error following the kth success can occur only if the kth success itself occurs as a result of guessing; that is, the subject necessarily is in state $\bar{C}$ when the kth success occurs. Letting g_k denote the probability that the kth success occurs by guessing, we can write the probability distribution

$$\Pr(\mathbf{J}_k = i) = \begin{cases} 1 - \alpha g_k & \text{for} \quad i = 0 \\ (1 - \alpha)\alpha^i g_k & \text{for} \quad i > 0, \end{cases} \tag{7}$$

where $\alpha = (1 - c)[1 - (1/r)]$. To obtain $\Pr(\mathbf{J}_k = 0)$, we note that 0 errors can occur in one of three ways: (1) the kth success occurs because the subject is in state C (which has probability $1 - g_k$) and necessarily a correct response occurs on the next trial; (2) the kth success occurs by guessing, the subject remaining in state $\bar{C}$ and again guessing correctly on the next trial [which has probability $g_k(1 - c)(1/r)$]; or (3) the kth success occurs by guessing but conditioning is effective on the trial (which has probability $g_k c$). Thus $\Pr(\mathbf{J}_k = 0) = 1 - g_k + g_k(1 - c)(1/r) + g_k c = 1 - \alpha g_k$. The event of i errors $(i > 0)$ between the kth and $(k + 1)$st successes can occur in one of two ways: (1) the kth and $(k + 1)$st successes occur by guessing $\{$with probability $g_k(1 - c)^{i+1}[1 - (1/r)]^i(1/r)\}$, or (2) the kth success occurs by guessing and conditioning does not take place until the trial immediately preceding the $(k + 1)$st success {with probability $g_k(1 - c)^i[1 - (1/r)]^i c$}. Hence

$$\begin{aligned}\Pr(\mathbf{J}_k = i) &= g_k(1 - c)^{i+1}\left(1 - \frac{1}{r}\right)^i \frac{1}{r} + g_k(1 - c)^i\left(1 - \frac{1}{r}\right)^i c \\ &= g_k\left(1 - \frac{1}{r}\right)^i (1 - c)^i\left[c + \frac{1}{r}(1 - c)\right] \\ &= g_k\alpha^i(1 - \alpha).\end{aligned}$$

From Eq. 7 we may obtain the mean and variance of $\mathbf{J}_k$, namely

$$E(\mathbf{J}_k) = \sum_{i=0}^{\infty} i \Pr(\mathbf{J}_k = i) = \frac{\alpha g_k}{1 - \alpha}, \tag{8}$$

and

$$\begin{aligned}\operatorname{Var}(\mathbf{J}_k) &= \sum_{i=0}^{\infty} i^2 \Pr(\mathbf{J}_k = i) - E(\mathbf{J}_k)^2 \\ &= \frac{\alpha g_k(1 + \alpha)}{(1 - \alpha)^2} - \frac{\alpha^2 g_k^{\,2}}{(1 - \alpha)^2} \\ &= \frac{\alpha g_k}{(1 - \alpha)^2}[1 + \alpha(1 - g_k)].\end{aligned} \tag{9}$$

In order to evaluate these quantities, we require an expression for g_k. Consider g_1, the probability that the first success will occur by guessing. It could occur in one of the following ways: (1) the subject guesses correctly on trial 1 (with probability $1/r$); (2) the subject guesses incorrectly on trial 1, conditioning does not occur, and the subject guesses successfully on trial 2 {this joint event has probability $[1 - (1/r)](1 - c)(1/r)$}; or (3) conditioning does not occur on trials 1 and 2, and the subject guesses incorrectly on both of these trials but guesses correctly on trial 3 {with probability $[1 - (1/r)]^2(1 - c)^2(1/r)$}, and so forth. Thus

$$\begin{aligned}g_1 &= \frac{1}{r} + \left(1 - \frac{1}{r}\right)(1 - c)\frac{1}{r} + \left(1 - \frac{1}{r}\right)^2(1 - c)^2\frac{1}{r} + \dots \\ &= \frac{1}{r}\sum_{i=0}^{\infty}\left(1 - \frac{1}{r}\right)^i(1 - c)^i \\ &= \frac{1}{(1 - \alpha)r}.\end{aligned}$$

Now consider the probability that the kth success occurs by guessing for $k > 1$. In order for this event to occur it must be the case that (1) the $(k - 1)$st success occurs by guessing, (2) conditioning fails to occur on the trial of the $(k - 1)$st success, and (3) since the subject is assumed to be in state $\bar{C}$ on the trial following the $(k - 1)$st success, the next correct response occurs by guessing, which has probability g_1. Hence

$$g_k = g_{k-1}(1 - c)g_1.$$

Solving this difference equation[3] we obtain

$$g_k = (1 - c)^{k-1}g_1^k.$$

Finally, substituting the expression obtained for g_1 yields

$$g_k = \frac{(1 - c)^{k-1}}{(r - \alpha r)^k}. \tag{10}$$

We may now combine Eqs. 7 and 10, inserting our original estimate of c, to obtain predictions about the number of errors between the kth and $(k + 1)$st success in Bower's data. To illustrate, for $k = 1$, the predicted mean is 0.361 and the observed value is 0.350.

To conclude our analysis of this model, we consider the probability p_k that a response sequence to a stimulus item will exhibit the property of no errors following the kth success. This event can occur in one of two ways: (1) the kth success occurs when the subject is in state C (the probability of which is $1 - g_k$), or (2) the kth success occurs when the subject is in state $\bar{C}$ and no errors occur on subsequent trials. Let b denote the probability of no more errors following a correct guess. Then

$$\begin{aligned} p_k &= (1 - g_k) + g_k b \\ &= 1 - g_k(1 - b). \end{aligned} \tag{11}$$

But the probability of no more errors following a successful guess is simply

$$\begin{aligned} b &= c + (1 - c)\frac{1}{r}c + (1 - c)^2\left(\frac{1}{r}\right)^2 c + \dots \\ &= \frac{c}{\alpha + c}. \end{aligned}$$

Substituting this result for b into Eq. 11, along with our expression for g_k in Eq. 10, we obtain

$$p_k = 1 - \frac{\alpha(1 - c)^{k-1}}{(\alpha + c)(r - \alpha r)^k}. \tag{12}$$

Observed and predicted values of p_k for Bower's experiment are shown in Table 2.

[3] The solution of this equation can quickly be obtained. Note that $g_2 = g_1(1 - c)g_1 = (1 - c)g_1^2$. Similarly, $g_3 = g_2(1 - c)g_1$; substituting the result for g_2, we obtain $g_3 = (1 - c)g_1^2(1 - c)g_1 = (1 - c)^2g_1^3$. If we continue in this fashion, it will be obvious that $g_k = (1 - c)^{k-1}g_1^k$.

We shall not pursue more consequences of this model.[4] The particular results we have examined were selected because they illustrated fundamental features of the model and also introduced mathematical techniques that will be needed later. In Bower's paper more than 30 predictions of the type presented here were tested, with results comparable to those exhibited above. The goodness-of-fit of theory to data in these instances is quite

Table 2 Observed and Predicted Values for p_k, the Probability of No Errors Following the kth Success

k	Observed p_k	Predicted p_k
0	0.255	0.256
1	0.628	0.636
2	0.812	0.822
3	0.869	0.912
4	0.928	0.957
5	0.963	0.979
6	0.973	0.990
7	0.990	0.995
8	0.990	0.997
9	0.993	0.998
10	0.996	0.999
11	1.000	1.000

(Interpret p_0 as the probability of no errors at all during the course of learning).

representative of what we may now expect to obtain routinely in simple learning experiments when experimental conditions have been appropriately arranged to approximate the simplifying assumptions of the mathematical model.

Concepts of the sort developed in this section can be extended to more traditional types of verbal learning situations involving stimulus similarity, meaningfulness, and the like. For example, Atkinson (1957) has presented a model for rote serial learning which is based on similar ideas and deals

[4] Bower also has compared the one-element model with a comparable single-operator linear model presented by Bush and Sternberg (1959). The linear model assumes that the probability of an incorrect response on trial n is a fixed number p_n, where $p_{n+1} = (1 - c)p_n$ and $p_1 = [1 - (1/r)]$. The one-element model and the linear model generate many identical predictions (e.g., mean learning curve), and it is necessary to look at the finer structure of the data to differentiate models. Among the 20 possible comparisons Bower makes between the two models, he finds that the one-element model comes closer to the data on 18.

with such variables as intertrial interval, list length, and types of errors (perseverative, anticipatory, or response-failure). Unfortunately, theoretical analyses of this sort for traditional experimental routines often lead to extremely complicated mathematical models with the result that only a few consequences of the axioms can be derived. Stated differently, a set of concepts may be general in terms of the range of situations to which it is applicable; nevertheless, in order to provide rigorous and detailed tests of these concepts, it is frequently necessary to contrive special experimental routines in which the theoretical analyses generate tractable mathematical systems.

1.3 Probabilistic Reinforcement Schedules

We shall now examine a one-element model for some simple two-choice learning problems. The one-element model for this situation, as contrasted with the paired-associate model, generates some predictions of behavior that are quite unrealistic, and for this reason we defer an analysis of experimental data until we consider comparable multi-element processes. The reason for presenting the one-element model is that it represents a convenient introduction to multi-element models and permits us to develop some mathematical tools in a simple fashion. Further, when we do discuss multi-element models, we shall employ a rather restrictive set of conditioning axioms. However, for the one-element model we may present an extremely general set of conditioning assumptions without getting into too much mathematical complexity. Therefore the analysis of the one-element case will suggest lines along which the multi-element models can be generalized.

The reference experiment (see, e.g., Estes & Straughan, 1954; Suppes & Atkinson, 1960) involves a long series of discrete trials. Each trial is initiated by the onset of a signal. To the signal the subject is required to make one of two responses which we denote A_1 and A_2. The trial is terminated with an E_1 or E_2 reinforcing event; the occurrence of E_i indicates that response A_i was the correct response for that trial. Thus in a human learning situation the subject is required on each trial to predict the reinforcing event he expects will occur by making the appropriate response—an A_1 if he expects E_1 and an A_2 if he expects E_2; at the end of the trial he is permitted to observe which event actually occurred. Initially the subject may have no preference between responses, but as information accrues to him over trials his pattern of choices undergoes systematic changes. The role of a model is to predict the detailed features of these changes.

The experimenter may devise various schedules for determining the sequence of reinforcing events over trials. For example, the probability of an E_1 may be (1) some function of the trial number, (2) dependent on previous responses of the subject, (3) dependent on the previous sequence of reinforcing events, or (4) some combination of the foregoing. For simplicity we consider a *noncontingent* reinforcement schedule. The case is defined by the condition that the probability of E_1 is constant over trials and independent of previous responses and reinforcements. It is customary in the literature to call this probability π; thus $\Pr(E_{1,n}) = \pi$ for all n. Here we are denoting by $E_{i,n}$ the event that reinforcement E_i occurs on trial n. Similarly, we shall represent by $A_{i,n}$ the event that response A_i occurs on trial n.

We assume that the stimulus situation comprising the signal light and the context in which it occurs can be represented theoretically by a single stimulus element that is sampled with probability 1 when the signal occurs. At the start of a trial the element is in one of three conditioning states: in state C_1 the element is conditioned to the A_1-response and in state C_2 to the A_2-response; in state C_0 the element is not conditioned to A_1 or to A_2. The response rules are similar to those presented earlier. When the subject is in C_1 or C_2, the A_1- or A_2-response occurs with probability 1. In state C_0 we assume that either response will be elicited equiprobably; that is, $\Pr(A_{1,n} \mid C_{0,n}) = \frac{1}{2}$. For some subjects a response bias may exist that would require the assumption $\Pr(A_{1,n} \mid C_{0,n}) = \beta$, where $\beta \neq \frac{1}{2}$. For these subjects it would be necessary to estimate β when applying the model. However, for simplicity we shall pursue only the case in which responses are equiprobable when the subject is in C_0.

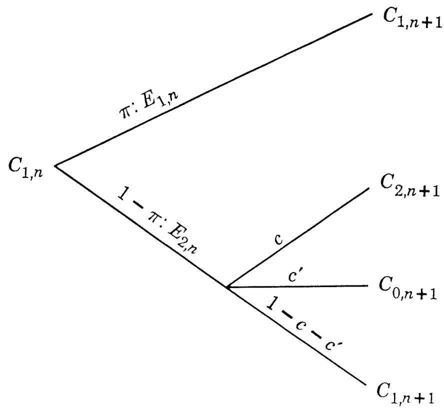

Fig. 2. Branching process, starting from state C_1 on trial n, for a one-element model in a two-choice, noncontingent case.

We now present a general set of rules governing changes in conditioning states. As the model is developed it will become obvious that for some experimental problems restrictions that greatly simplify the process can be imposed.

If the subject is in state C_1 and an E_1 occurs (i.e., the subject makes an A_1-response, which is correct), then he will remain in C_1. However, if the subject is in C_1 and an E_2 occurs, then with probability c the subject goes to C_2 and with probability c' to C_0. Comparable rules apply when the subject is in C_2. Thus, if the subject is in C_1 or C_2 and his response is correct, he will remain in C_1 or C_2. If, however, he is in C_1 or C_2 and his response is not correct, then he may shift to one of the other conditioning states, which reduces the probability of repeating the same response on the next trial.

Finally, if the subject is in C_0 and an E_1 or E_2 occurs, then with probability c'' the subject moves to C_1 or C_2, respectively.[5] Thus, to summarize, for $i, j = 1, 2$ and $i \neq j$,

$$\begin{aligned} \Pr(C_{i,n+1} \mid E_{i,n}C_{i,n}) &= 1 \\ \Pr(C_{0,n+1} \mid E_{j,n}C_{i,n}) &= c' \\ \Pr(C_{j,n+1} \mid E_{j,n}C_{i,n}) &= c \\ \Pr(C_{i,n+1} \mid E_{i,n}C_{0,n}) &= c'' \end{aligned} \tag{13}$$

where $0 < c'' \leqslant 1$ and $0 < c + c' \leqslant 1$.

We now use the assumptions of the preceding paragraphs and the particular assumptions for the noncontingent case to derive the transition matrix in the conditioning states. In making such a derivation it is convenient to represent the various *possible* occurrences on a trial by a tree. Each set of branches emanating from a point represents a mutually exclusive and exhaustive set of possibilities. For example, suppose that at the start of trial n the subject is in state C_1; the tree in Fig. 2 represents the possible changes that can occur in the conditioning state.

[5] Here we assume that the subject's response does not affect the change; that is, if the subject is in C_0 and an E_1 occurs, then he will move to C_1 with probability c'', no matter whether A_1 or A_2 has occurred. This assumption is not necessary and we could readily have the actual response affect change. For example, we might postulate c_1'' for an A_1E_1 or A_2E_2 combination, and c_2'' for the A_1E_2 or A_2E_1 combination; that is,

$$\Pr(C_{1,n+1} \mid E_{1,n}A_{1,n}C_{0,n}) = \Pr(C_{2,n+1} \mid E_{2,n}A_{2,n}C_{0,n}) = c_1''$$

and

$$\Pr(C_{1,n+1} \mid E_{1,n}A_{2,n}C_{0,n}) = \Pr(C_{2,n+1} \mid E_{2,n}A_{1,n}C_{0,n}) = c_2''$$

where

$$c_1'' \neq c_2''.$$

However, such additions make the mathematical process more complicated and should be introduced only when the data clearly require them.

The first set of branches is associated with the reinforcing event on trial n. If the subject is in C_1 and an E_1 occurs, then he will stay in state C_1 on the next trial. However, if an E_2 occurs, then with probability c he will go to C_2, with probability c' he will go to C_0, and with probability $1 - c - c'$ he will remain in C_1.

Each path of a tree, from a beginning point to a terminal point, represents a possible outcome on a given trial. The probability of each path is obtained by multiplying the appropriate conditional probabilities. Thus for the tree in Fig. 2 the probability of the bottom path may be represented by $\Pr(E_{2,n} \mid C_{1,n}) \Pr(C_{1,n+1} \mid E_{2,n}C_{1,n}) = (1 - \pi)(1 - c - c')$. Two of the four paths lead from C_1 to C_1; hence

$$p_{11} = \Pr(C_{1,n+1} \mid C_{1,n}) = \pi + (1 - \pi)(1 - c - c').$$

Similarly, $p_{10} = (1 - \pi)c'$ and $p_{12} = (1 - \pi)c$, where p_{ij} denotes the probability of a one-step transition from C_i to C_j.

For the C_0 state we have the tree given in Fig. 3. On the top branch an E_1 event is indicated; by Eq. 13 the probability of going to C_1 is c'' and of staying in C_0 is $1 - c''$. A similar analysis holds for the bottom branches. Thus we have

$$\begin{aligned} p_{01} &= \pi c'' \\ p_{02} &= (1 - \pi)c'' \\ P_{00} &= 1 - c''. \end{aligned}$$

A combination of these results and the comparable results for C_2 yields the following transition matrix:

$$P = \begin{array}{c} \\ C_1 \\ C_0 \\ C_2 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_0 & C_2 \end{array} \\ \begin{bmatrix} 1 - (1 - \pi)(c' + c) & c'(1 - \pi) & c(1 - \pi) \\ c''\pi & 1 - c'' & c''(1 - \pi) \\ c\pi & c'\pi & 1 - \pi(c' + c) \end{bmatrix} \end{array}. \qquad (14)$$

As in the case of the paired-associate model, a large number of predictions can be derived easily for this process. However, we shall select only a few that will help to clarify the fundamental properties of the model. We begin by considering the asymptotic probability of a particular conditioning state and, in turn, the asymptotic probability of an A_1-response. The following notation will prove useful: let $[p_{ij}]$ be the transition matrix and define $p_{ij}^{(n)}$ as the probability of being in state j on trial $r + n$, given that at trial r the subject was in state i. The quantity is defined recursively:

$$p_{ij}^{(1)} = p_{ij}, \qquad p_{ij}^{(n+1)} = \sum_v p_{iv}p_{vj}^{(n)}.$$

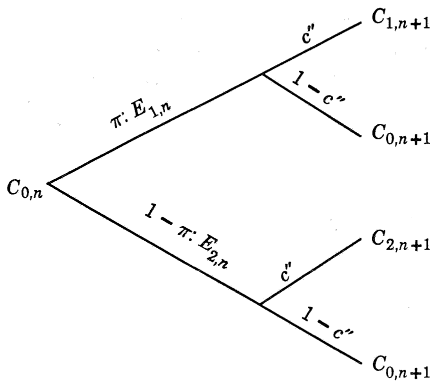

Fig. 3. Branching process, starting from state C_0 on trial n, for a one-element model in a two-choice, noncontingent case.

Moreover, if the appropriate limit exists and is independent of i, we set

$$u_j = \lim_{n\to\infty} p_{ij}^{(n)}.$$

The limiting quantities u_j exist for any finite-state Markov chain that is irreducible and aperiodic. A Markov chain is irreducible if there is no closed proper subset of states; that is, no proper subset of states such that once within this set the probability of leaving it is 0. For example, the chain whose transition matrix is

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \end{array}$$

is reducible because the set $\{1, 2\}$ of states is a proper closed subset. A Markov chain is aperiodic if there is no fixed period for return to any state and periodic if a return to some initial state j is impossible except at t, $2t$, $3t$, . . . trials for $t > 1$. Thus the chain whose matrix is

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

has period $t = 3$ for return to each state.

If there are r states, we call the vector $\mathbf{u} = (u_1, u_2, \ldots, u_r)$ the *stationary probability* vector of the chain. It may be shown (Feller, 1957; Kemeny & Snell, 1959) that the components of this vector are the solutions of the r linear equations

$$\begin{aligned} u_1 &= \sum_{v=1}^{r} u_v p_{v1} \\ u_2 &= \sum_{v=1}^{r} u_v p_{v2} \\ &\;\vdots \\ u_r &= \sum_{v=1}^{r} u_v p_{vr} \end{aligned} \tag{15}$$

such that $\sum_{v=1}^{r} u_v = 1$. Thus, to find the asymptotic probabilities u_j of the states, we need find only the solution of the r equations. The intuitive basis of this system of equations seems clear. Consider a two-state chain. Then the probability p_{n+1} of being in state 1 on trial $n + 1$ is the probability of being in state 1 on trial n and going to 1 plus the probability of being in state 2 on trial n and going to 1; that is

$$p_{n+1} = p_{11}p_n + p_{21}(1 - p_n).$$

But at asymptote $p_{n+1} = p_n = u_1$ and $1 - p_n = u_2$, whence

$$u_1 = p_{11}u_1 + p_{21}u_2,$$

which is the first of the two equations of the system when $r = 2$.

It is clear that the chain represented by the matrix P of Eq. 14 is irreducible and aperiodic; thus the asymptotes exist and are independent of the initial probability distribution on the states. Let $[p_{ij}]$ $(i, j = 1, 2, 3)$ be any 3×3 transition matrix. Then we seek the numbers u_j such that $u_j = \sum_v u_v p_{vj}$ and $\Sigma\, u_j = 1$. The general solution is given by $u_j = D_j/D$, where

$$\begin{aligned} D_1 &= p_{31}(1 - p_{22}) + p_{21}p_{32} \\ D_2 &= p_{31}p_{12} + p_{32}(1 - p_{11}) \\ D_3 &= (1 - p_{11})(1 - p_{22}) - p_{21}p_{12} \\ D &= D_1 + D_2 + D_3. \end{aligned} \tag{16}$$

Inserting in these equations the equivalents of the p_{ij} from the transition matrix and renumbering the states appropriately, we obtain

$$\begin{aligned} D_1 &= \pi c''(c + c'\pi) \\ D_0 &= \pi(1 - \pi)c'(c' + 2c) \\ D_2 &= (1 - \pi)c''[c + c'(1 - \pi)]. \end{aligned}$$

Since D is the sum of the D_j's and since $u_j = D_j/D$, we may divide the numerator and denominator by $(c'')^2$ and obtain

$$\begin{aligned} u_1 &= \frac{\pi(\rho + \epsilon\pi)}{\pi(\rho + \epsilon\pi) + \pi(1 - \pi)\epsilon(\epsilon + 2\rho) + (1 - \pi)[\rho + \epsilon(1 - \pi)]} \\ u_0 &= \frac{\pi(1 - \pi)\epsilon(\epsilon + 2\rho)}{\pi(\rho + \epsilon\pi) + \pi(1 - \pi)\epsilon(\epsilon + 2\rho) + (1 - \pi)[\rho + \epsilon(1 - \pi)]} \\ u_2 &= 1 - u_1 - u_0, \end{aligned} \tag{17}$$

where $\rho = c/c''$ and $\epsilon = c'/c''$.

By our response axioms we have

$$\Pr(A_{1,n}) = \Pr(C_{1,n}) + \tfrac{1}{2}\Pr(C_{0,n})$$

for all n. Hence

$$\begin{aligned} \lim_{n\to\infty} \Pr(A_{1,n}) &= u_1 + \tfrac{1}{2}u_0 \\ &= \frac{\pi(\rho + \epsilon\rho + \frac{1}{2}\epsilon^2) + \pi^2(\epsilon - \epsilon\rho - \frac{1}{2}\epsilon^2)}{\pi(\epsilon^2 + 2\epsilon\rho - 2\epsilon) + \pi^2(2\epsilon - \epsilon^2 - 2\epsilon\rho) + \rho + \epsilon}. \end{aligned} \tag{18}$$

An inspection of Eq. 18 indicates that the asymptotic probability of an A_1-response is a function of π, ρ, and ϵ. As will become clear later, the value of $\Pr(A_{1,\infty})$ is bounded in the open interval from $\frac{1}{2}$ to $\pi^2/[\pi^2 + (1 - \pi)^2]$; whether $\Pr(A_{1,\infty})$ is above or below π depends on the values of ρ and ϵ.

We now consider two special cases of our one-element model. The first case is comparable to the multi-element models to be discussed later, whereas the second case is, in some respects, the complement of the first case.

Case of $c' = 0$. Let us rewrite Eq. 14 with $c' = 0$. Then the transition matrix has the following canonical form:

$$P = \begin{array}{c} \\ C_1 \\ C_2 \\ C_0 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_0 \end{array} \\ \begin{bmatrix} 1 - c(1 - \pi) & c(1 - \pi) & 0 \\ c\pi & 1 - c\pi & 0 \\ c''\pi & c''(1 - \pi) & 1 - c'' \end{bmatrix} \end{array}. \tag{19}$$

We note that once the subject has left state C_0 he can never return. In fact, it is obvious that $\Pr(C_{0,n}) = \Pr(C_{0,1})(1 - c'')^{n-1}$ where $\Pr(C_{0,1})$ is the initial probability of being in C_0. Thus, except on early trials, C_0 is not part of the process, and the subject in the long run fluctuates between C_1 and C_2, being in C_1 on a proportion π of the trials.

From Eq. 19 we have also

$$\Pr(C_{1,n+1}) = \Pr(C_{1,n})[1 - c(1 - \pi)] + \Pr(C_{2,n})c\pi + \Pr(C_{0,n})c''\pi;$$

that is, the probability of being in C_1 on trial $n + 1$ is equal to the probability of being in C_1 on trial n times the probability p_{11} of going from C_1 to C_1 plus the probability of being in C_2 times p_{21} plus the probability of being in C_0 times p_{01}. For simplicity let $x_n = \Pr(C_{1,n})$, $y_n = \Pr(C_{2,n})$, and $z_n = \Pr(C_{0,n})$. Now we know that $z_n = z_1(1 - c'')^{n-1}$ and also that $x_n + y_n + z_n = 1$, or $y_n = 1 - x_n - z_1(1 - c'')^{n-1}$. Making these substitutions in the foregoing recursion yields

$$\begin{aligned} x_{n+1} &= x_n[1 - c(1 - \pi)] + z_1 c''\pi(1 - c'')^{n-1} + c\pi[1 - x_n - z_1(1 - c'')^{n-1}] \\ &= x_n(1 - c) + z_1(1 - c'')^{n-1}\pi(c'' - c) + c\pi. \end{aligned}$$

This difference equation has the following solution[6]:

$$x_n = \pi - (\pi - x_1)(1 - c)^{n-1} - \pi z_1[(1 - c'')^{n-1} - (1 - c)^{n-1}].$$

But $\Pr(A_{1,n}) = x_n + \frac{1}{2}z_n$; hence

$$\begin{aligned} \Pr(A_{1,n}) = \pi - [\pi - \pi \Pr(C_{0,1}) - \Pr(C_{1,1})](1 - c)^{n-1} \\ - \Pr(C_{0,1})(\pi - \tfrac{1}{2})(1 - c'')^{n-1}. \quad (20) \end{aligned}$$

If $\Pr(C_{0,1}) = 0$, then we have a simple exponential learning function starting at $\Pr(C_{1,1})$ and approaching π at a rate determined by c. If $\Pr(C_{0,1}) \neq 0$, then the rate of approach is a function of both c and c''.

We now consider one simple sequential prediction to illustrate another feature of the one-element model for $c' = 0$. Specifically, consider the probability of an A_1-response on trial $n + 1$ given a reinforced A_1-response on trial n; namely $\Pr(A_{1,n+1} \mid E_{1,n}A_{1,n})$. Note first of all that

$$\Pr(A_{1,n+1} \mid E_{1,n}A_{1,n}) \Pr(E_{1,n}A_{1,n}) = \Pr(A_{1,n+1}E_{1,n}A_{1,n}).$$

[6] The solution of such a difference equation can readily be obtained. Consider $x_{n+1} = ax_n + bc^{n-1} + d$ where a, b, c, and d are constants. Then

(1) $$x_2 = ax_1 + b + d.$$

Similarly, $x_3 = ax_2 + bc + d$ and substituting (1) for x_2 we obtain

(2) $$x_3 = a^2x_1 + ab + ad + bc + d.$$

Similarly, $x_4 = ax_3 + bc^2 + d$ and substituting (2) for x_3 we obtain

(3) $$x_4 = a^3x_1 + a^2b + a^2d + abc + ad + bc^2 + d.$$

If we continue in this fashion, it will be obvious that for $n \geqslant 2$

$$x_n = a^{n-1}x_1 + d\sum_{i=0}^{n-2} a^i + a^{n-2}b\sum_{i=0}^{n-2}\left(\frac{c}{a}\right)^i.$$

Carrying out the summations yields the desired results. See Jordan (1950, pp. 583–584) for a detailed treatment.

Further, we may write

$$
\begin{aligned}
&\Pr(A_{1,n+1}E_{1,n}A_{1,n}) \\
&\quad = \sum_{i,j} \Pr(A_{1,n+1}C_{i,n+1}E_{1,n}A_{1,n}C_{j\ n}) \\
&\quad = \sum_{i,j} \Pr(A_{1,n+1} \mid C_{i,n+1}E_{1,n}A_{1,n}C_{j\ n}) \Pr(C_{i,n+1} \mid E_{1,n}A_{1,n}C_{j,n}) \\
&\qquad\qquad \cdot \Pr(E_{1,n} \mid A_{1,n}C_{j\ n}) \Pr(A_{1,n} \mid C_{j,n}) \Pr(C_{j,n}).
\end{aligned}
$$

By assumption the probability of a response is determined solely by the conditioning state, hence

$$\Pr(A_{1,n+1} \mid C_{i,n+1}E_{1,n}A_{1,n}C_{j,n}) = \Pr(A_{1,n+1} \mid C_{i,n+1}).$$

Further, by assumption, the probability of an E_1-event is independent of other events, and $\Pr(E_{1,n} \mid A_{1,n}C_{j,n}) = \pi$. Substituting these results in the foregoing expression, we obtain

$$
\begin{aligned}
\Pr(A_{1,n+1}E_{1,n}A_{1,n}) = \pi \sum_{i,j} \Pr(A_{1,n+1} \mid C_{i,n+1}) \Pr(C_{i,n+1} \mid E_{1,n}A_{1,n}C_{j,n}) \\
\cdot \Pr(A_{1,n} \mid C_{j,n}) \Pr(C_{j,n}).
\end{aligned}
$$

Both i and j run over 0, 1, and 2, and therefore there are nine terms in the sum; but note that when $i = 2$, the term $\Pr(A_{1,n+1} \mid C_{i,n+1})$ is zero and when $j = 2$ the term $\Pr(A_{1,n} \mid C_{j,n})$ is zero. Consequently it suffices to limit i and j to 0 and 1, and we have

$$
\begin{aligned}
&\Pr(A_{1,n+1}E_{1,n}A_{1,n}) \\
&= \pi \sum_{i=0}^{1} \Pr(A_{1,n+1} \mid C_{i,n+1}) \Pr(C_{i,n+1} \mid E_{1,n}A_{1,n}C_{1,n}) \Pr(A_{1,n} \mid C_{1,n}) \Pr(C_{1,n}) \\
&+ \pi \sum_{i=0}^{1} \Pr(A_{1,n+1} \mid C_{i,n+1}) \Pr(C_{i,n+1} \mid E_{1,n}A_{1,n}C_{0,n}) \Pr(A_{1,n} \mid C_{0,n}) \Pr(C_{0,n}).
\end{aligned}
$$

Since the subject cannot leave state C_1 on a trial when A_1 is reinforced, we know that

$$\Pr(C_{1,n+1} \mid E_{1,n}A_{1,n}C_{1,n}) = 1 \quad \text{and} \quad \Pr(C_{0,n+1} \mid E_{1,n}A_{1,n}C_{1,n}) = 0;$$

further, $\Pr(A_{1,n+1} \mid C_{1,n+1}) = 1$. Therefore the first sum is simply $\pi \Pr(C_{1,n})$. For the second sum, $\Pr(C_{1,n+1} \mid E_{1,n}A_{1,n}C_{0,n}) = c''$ and $\Pr(C_{0,n+1} \mid E_{1,n}A_{1,n}C_{0,n}) = 1 - c''$. Further, $\Pr(A_{1,n} \mid C_{0,n}) = \frac{1}{2}$; hence for the second sum we obtain

$$\pi[c''\tfrac{1}{2} + \tfrac{1}{2}(1 - c'')\tfrac{1}{2}] \Pr(C_{0,n}).$$

Combining these results,

$$\Pr(A_{1\ n+1}E_{1,n}A_{1,n}) = \pi\{\Pr(C_{1,n}) + \tfrac{1}{2}\Pr(C_{0,n})[c'' + (1 - c'')\tfrac{1}{2}]\}.$$

But

$$\Pr(E_{1,n}A_{1,n}) = \Pr(E_{1,n} \mid A_{1,n}) \Pr(A_{1,n}) = \pi \Pr(A_{1,n}),$$

whence

$$\Pr(A_{1,n+1} \mid E_{1,n}A_{1,n}) = \frac{\Pr(C_{1,n}) + \frac{1}{2}\Pr(C_{0,n})[c'' + (1 - c'')\frac{1}{2}]}{\Pr(A_{1,n})}.$$

We know that $\Pr(C_{1,n})$ and $\Pr(A_{1,n})$ both approach π in the limit and that $\Pr(C_{0,n})$ approaches 0. Therefore we predict that

$$\lim_{n\to\infty} \Pr(A_{1,n+1} \mid E_{1,n}A_{1,n}) = 1.$$

This prediction provides a sharp test for this particular case of the model and one that is certain to fail in almost any experimental situation; that is, even after a large number of trials it is hard to conceive of an experimental procedure such that a response will be repeated with probability 1 if it occurred and was reinforced on the preceding trial. Later we shall consider a multi-element model that provides an excellent description of many sets of data but is based on essentially the same conditioning rules specified by this case of $c' = 0$. It should be emphasized that deterministic predictions of the sort given in the foregoing equation are peculiar to one-element models; for the multi-element case such difficulties do not arise. This point is amplified later.

Case of $c = 0$. We now consider the case in which direct counterconditioning does not occur, that is, $c = 0$, and thus $\rho = 0$ and $0 < \epsilon < \infty$. With this restriction the chain is still ergodic, since it is possible to go from every state to every other state, but transitions between C_1 and C_2 must go by way of C_0. Letting $\rho = 0$ in Eq. 18, we obtain

$$\Pr(A_{1,\infty}) = \frac{\pi^2 + \frac{1}{2}\pi(1 - \pi)\epsilon}{\pi^2 + \pi(1 - \pi)\epsilon + (1 - \pi)^2}. \tag{21}$$

From Eq. 21 we can draw some interesting conclusions about the relationship of the asymptotic response probabilities to the ratio $\epsilon = c'/c''$. Differentiating with respect to ϵ, we obtain

$$\frac{\partial}{\partial\epsilon}\Pr(A_{1,\infty}) = \frac{\pi(1 - \pi)(\frac{1}{2} - \pi)}{[\pi^2 + (1 - \pi)^2 + \pi(1 - \pi)\epsilon]^2}.$$

If $\pi(1 - \pi)(\frac{1}{2} - \pi) \neq 0$, then $\Pr(A_{1,\infty})$ has no maximum for ϵ in the open interval $(0, \infty)$, which is the permissible range on ϵ. In fact, since the sign of the derivative is independent of ϵ, we know that $\Pr(A_{1,\infty})$ is either monotone increasing or monotone decreasing in ϵ: strictly increasing if $\pi(1 - \pi)(\frac{1}{2} - \pi) > 0$ (i.e., $\pi < \frac{1}{2}$) and decreasing if $\pi(1 - \pi)(\frac{1}{2} - \pi) < 0$ (i.e., $\pi > \frac{1}{2}$). Moreover, because of the monotonicity of $\Pr(A_{1,\infty})$ in ϵ, it is easy to compute bounds from Eq. 21. First, we see immediately that the lower bound (assuming $\pi > \frac{1}{2}$) is $\lim_{\epsilon \to \infty} \Pr(A_{1,\infty}) = \frac{1}{2}$. Second, when ϵ is very small, $\Pr(A_{1,\infty})$ approaches $\pi^2/[\pi^2 + (1 - \pi)^2]$. Note, however, that Eq. 21 is inapplicable when $\epsilon = 0$; for if both $c = 0$ and $c' = 0$ the transition matrix (Eq. 14) reduces to

$$P = \begin{bmatrix} 1 & 0 & 0 \\ c''\pi & 1 - c'' & c''(1 - \pi) \\ 0 & 0 & 1 \end{bmatrix},$$

and, if the process starts in C_0, $\Pr(A_{1,\infty}) = \pi$. But for $\epsilon > 0$, if $\pi > \frac{1}{2}$, $\Pr(A_{1,\infty})$ is a decreasing function of ϵ and its values lie in the half-open interval

$$\tfrac{1}{2} \leqslant \Pr(A_{1,\infty}) < \frac{\pi^2}{\pi^2 + (1 - \pi)^2}.$$

It is readily determined that probability matching would not generally be predicted in this case. When c'/c'' is greater than 2, the predicted value of $\Pr(A_{1,\infty})$ is less than π, and when this ratio is less than 2 the predicted value of $\Pr(A_{1,\infty})$ is greater than π.

Finally, we derive $\Pr(A_{1,n+1} \mid E_{1,n}A_{1,n})$ for this case. The derivation is identical to that given for $c' = 0$. Hence

$$\lim_{n \to \infty} \Pr(A_{1,n+1} \mid E_{1,n}A_{1,n}) = \frac{u_1 + \frac{1}{2}u_0[c'' + (1 - c'')\frac{1}{2}]}{u_1 + \frac{1}{2}u_0}.$$

Note, however, that for $c = 0$ the quantity u_0 is never 0 (except for $\pi = 0, 1$), and consequently $\Pr(A_{1,n+1} \mid E_{1,n}A_{1,n})$ is always less than 1.

Contingent Reinforcement. As a final example we shall apply the one-element model to a situation in which the reinforcing event on trial n is contingent on the response on that trial. Simple contingent reinforcement is defined by two probabilities π_{11} and π_{21} such that

$$\Pr(E_{1,n} \mid A_{1,n}) = \pi_{11} \quad \text{and} \quad \Pr(E_{1,n} \mid A_{2,n}) = \pi_{21}.$$

We consider the case of the model in which $c' = 0$ and $\Pr(C_{0,1}) = 0$; that is, the subject is not in state C_0 on trial 1 and (since $c' = 0$) he can never reach C_0 from C_1 or C_2. Hence on all trials he is in C_1 or C_2, and transitions between these states are governed by the single parameter c. The trees for the C_1 and C_2 states are given in Fig. 4.

The transition matrix is

$$P = \begin{array}{c} \\ C_1 \\ C_2 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{bmatrix} 1 - (1 - \pi_{11})c & (1 - \pi_{11})c \\ c\pi_{21} & 1 - c\pi_{21} \end{bmatrix} \end{array},$$

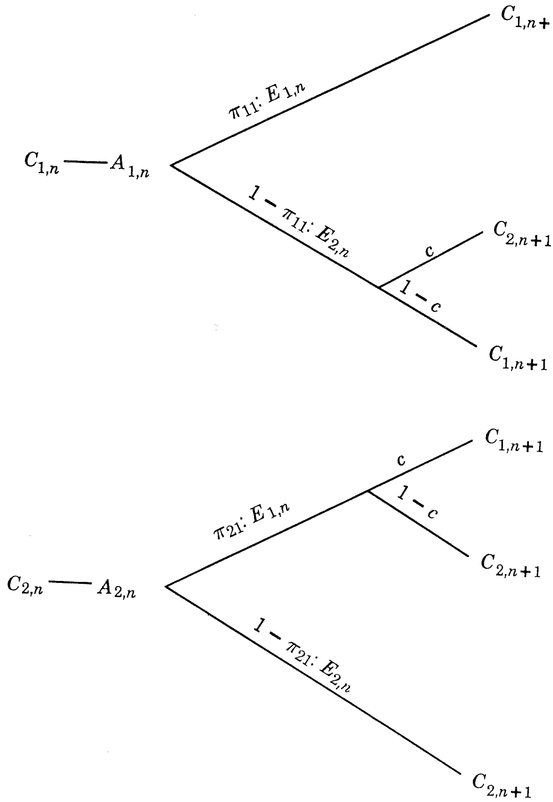

Fig. 4. Branching process for one-element model in two-choice, contingent case.

and, in terms of this matrix, we may write

$$\Pr(C_{1,n+1}) = \Pr(C_{1,n})[1 - (1 - \pi_{11})c] + \Pr(C_{2,n})c\pi_{21}.$$

But $\Pr(C_{2,n}) = 1 - \Pr(C_{1,n})$ and $\Pr(C_{1,n}) = \Pr(A_{1,n})$; hence

$$\Pr(A_{1,n+1}) = \Pr(A_{1,n})[1 - (1 - \pi_{11})c - c\pi_{21}] + c\pi_{21}.$$

This difference equation has the solution

$$\Pr(A_{1,n}) = \Pr(A_{1,\infty}) - [\Pr(A_{1,\infty}) - \Pr(A_{1,1})][1 - c(1 - \pi_{11} + \pi_{21})]^{n-1},$$

where

$$\Pr(A_{1,\infty}) = \frac{\pi_{21}}{1 - \pi_{11} + \pi_{21}}.$$

The asymptote is independent of c, and the rate of approach is determined by the quantity $c(1 - \pi_{11} + \pi_{21})$. It is interesting to note that the learning function for $\Pr(A_{1,n})$ in this case of the one-element model is identical to that of the linear model (cf. Estes & Suppes, 1959a).

2. MULTI-ELEMENT PATTERN MODELS

2.1 General Formulation

In the literature of stimulus sampling theory a variety of proposals has been made for conceptually representing the stimulus situation. Fundamental to all of these suggestions has been the distinction between pattern elements and component elements. For the one-element case this distinction does not play a serious role, but for multi-element formulations these alternative representations of the stimulus situation specify different mathematical processes.

In component models the stimulating situation is represented as a population of elements which the learner is viewed as sampling from trial to trial. It is assumed that the conditioning of individual elements to responses occurs independently as the elements are sampled in conjunction with reinforcing events and that the response probability in the presence of a sample containing a number of elements is determined by an averaging rule. The principal consideration has been to account for response variability to an apparently constant stimulus situation by postulating random fluctuations from trial to trial in the particular sample of stimulus elements affecting the learner. These component models have provided a mechanism for effecting a reconciliation between the picture of gradual change usually exhibited by the learning curve and the all-or-none law of association.

For many experimental situations a detailed account of the quantitative properties of learning can be given by component models that assume discrete associations between responses and the independently variable elements of a stimulating situation. However, in some cases predictions from component models fail, and it appears that a simple account of the learning process requires the assumption that responses become associated, not with separate components or aspects of a stimulus situation, but with total patterns of stimulation considered as units. The model presented in this section is intended to represent such a case. In it we assume that an experimentally specified stimulating situation can be conceived as an assemblage of distinct, mutually exclusive patterns of stimulation, each of which becomes conditioned to responses on an all-or-none basis. By "mutually exclusive" we mean that exactly one of the patterns occurs (is sampled by the subject) on each trial. By "distinct" we mean that no generalization occurs from one pattern to another. Thus the clearest experimental interpretation would involve a set of patterns having no common elements (i.e., common properties or components). In practice the pattern model has also been applied with considerable success to experiments in which the alternative stimuli have some common elements but nevertheless are sufficiently discriminable so that generalization effects (e.g., "confusion errors") are small and can be neglected without serious error.

In this presentation we shall limit consideration to cases in which patterns are sampled randomly with equal likelihood so that if there are N patterns each has probability $1/N$ of being sampled on a trial. This sampling assumption represents only one way of formulating the model and is presented here because it generates a fairly simple mathematical process and provides a good account of a variety of experimental results. However, this particular scheme for sampling patterns has restricted applicability. For example, in certain experiments it can be demonstrated that the stimulus array to which the subject responds is in large part determined by events on previous trials; that is, trace stimulation associated with previous responses and rewards determines the stimulus pattern to which the subject responds. When this is the case, it is necessary to postulate a more general rule for sampling patterns than the random scheme proposed (e.g., see the discussion of "hypothesis models" in Suppes & Atkinson, 1960).

Before stating the axioms for the pattern model to be considered in this section, we define the following notions. As before, the behaviors available to the subject are categorized into mutually exclusive and exhaustive response classes ($A_1, A_2, \ldots, A_r$). The possible experimenter-defined outcomes of a trial (e.g., giving or withholding reward, unconditioned stimulus, knowledge of results) are classified by their effect on response probability and are represented by a mutually exclusive and exhaustive set of reinforcing events ($E_0, E_1, \ldots, E_r$). The event E_i ($i \neq 0$) indicates that response A_i is reinforced and E_0 represents any trial outcome whose effect is neutral (i.e., reinforces none of the A_i's). The subject's response and the experimenter-defined outcomes are observable, but the occurrence of E_i is a purely hypothetical event that represents the reinforcing effect of the trial outcome. Event E_i is said to have occurred when the outcome of a trial increases the probability of response A_i in the presence of the given stimulus—provided, of course, that this probability is not already at its maximum value.

We now present the axioms. The first group of axioms deals with the conditioning of sampled patterns, the second group with the sampling of patterns, and the third group with responses.

Conditioning Axioms

C1. *On every trial each pattern is conditioned to exactly one response.*

C2. *If a pattern is sampled on a trial, it becomes conditioned with probability c to the response (if any) that is reinforced on the trial; if it is already conditioned to that response, it remains so.*

C3. *If no reinforcement occurs on a trial (i.e., E_0 occurs), there is no change in conditioning on that trial.*

C4. *Patterns that are not sampled on a trial do not change their conditioning on that trial.*

C5. *The probability c that a sampled pattern will be conditioned to a reinforced response is independent of the trial number and the preceding events.*

Sampling Axioms

S1. *Exactly one pattern is sampled on each trial.*

S2. *Given the set of N patterns available for sampling on a trial, the probability of sampling a given pattern is $1/N$, independent of the trial number and the preceding events.*

Response Axiom

R1. *On any trial that response is made to which the sampled pattern is conditioned.*

Later in this section we apply these axioms to a two-choice learning experiment and to a paired-comparison study. First, however, we shall prove several general theorems. Before we can begin our analysis it is necessary to define the notion of a conditioning state. For the axioms given, all patterns are sampled with equal probability, and it suffices to let the state of conditioning indicate the number of patterns conditioned to each response. Hence for r responses the conditioning states are the ordered r-tuples $\langle k_1, k_2, \ldots, k_r \rangle$ where $k_i = 0, 1, 2, \ldots, N$ and $k_1 + k_2 + \ldots + k_r = N$; the integer k_i denotes the number of patterns conditioned to the A_i response. The number of possible conditioning states is $\binom{N+r-1}{N}$. (In a generalized model, which permitted different patterns to have different likelihoods of being sampled, it would be necessary to specify not only the number of patterns conditioned to a response but also the sampling probabilities associated with the patterns.) For simplicity we limit consideration in this section to the case of two alternatives, except for one example in which $r = 3$. Given only two alternatives, we denote the conditioning state on trial n of an experiment as $C_{i,n}$, where $i = 0, 1, 2, \ldots, N$; the subscript i indicates the number of patterns conditioned to A_1 and $N - i$ the number conditioned to A_2.

TRANSITION PROBABILITIES. Only one pattern is sampled per trial; therefore the subject can go from state C_i only to one of the three states C_{i-1}, C_i, or C_{i+1} on any given trial. The probabilities of these transitions depend on the value of the conditioning parameter c, the reinforcement schedule, and the value of i. We now proceed to compute these probabilities.

If the subject is in state C_i on trial n and an E_1 occurs, then the possible outcomes are indicated by the tree in Fig. 5. On the upper main branch, which has probability i/N, a pattern that is conditioned to A_1 is sampled and, since an E_1-reinforcement occurs, the pattern remains conditioned to A_1. Hence the conditioning state on trial $n + 1$ is the same as on trial n (see Axiom C2). On the lower main branch, which has probability $(N - i)/N$, a pattern conditioned to A_2 is sampled; then with probability c the pattern is conditioned to A_1 and the subject moves to conditioning state C_{i+1}, whereas with probability $1 - c$ conditioning is not effective and the subject remains in state C_i. Putting these results together, we obtain

$$\Pr(C_{i+1,n+1} \mid E_{1,n}C_{i,n}) = c\,\frac{N-i}{N}$$
$$\Pr(C_{i,n+1} \mid E_{1,n}C_{i,n}) = 1 - c + c\,\frac{i}{N}. \tag{22a}$$

Similarly, if an E_2 occurs on trial n,

$$\Pr(C_{i-1,n+1} \mid E_{2,n}C_{i,n}) = c\,\frac{i}{N}$$
$$\Pr(C_{i,n+1} \mid E_{2,n}C_{i,n}) = 1 - c + c\,\frac{N-i}{N}. \tag{22b}$$

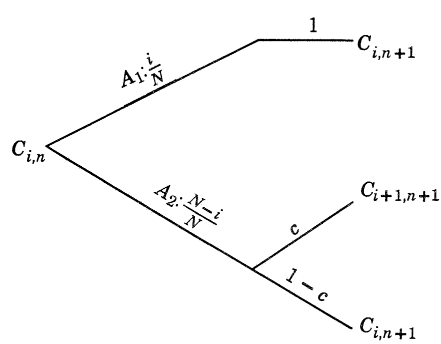

Fig. 5. Branching process for N-element model on a trial when the subject starts in state C_i and an E_1-event occurs.

By Axiom C3, if an E_0 occurs, then

$$\Pr(C_{i,n+1} \mid E_{0,n}C_{i,n}) = 1. \tag{22c}$$

Noting that a transition upward can occur only when a pattern conditioned to A_2 is sampled on an E_1-trial and a transition downward can occur only when a pattern conditioned to A_1 is sampled on an E_2-trial, we can combine the results from Eq. 22*a*-*c* to obtain

$$\Pr(C_{i+1,n+1} \mid C_{i,n}) = c\,\frac{N-i}{N}\Pr(E_{1,n} \mid A_{2,n}C_{i,n}) \tag{23a}$$

$$\Pr(C_{i-1,n+1} \mid C_{i,n}) = c\,\frac{i}{N}\Pr(E_{2,n} \mid A_{1,n}C_{i,n}) \tag{23b}$$

$$\begin{aligned}\Pr(C_{i,n+1} \mid C_{i,n}) = 1 - c + c\Big[\frac{i}{N}&\Pr(E_{1,n} \mid A_{1,n}C_{i,n})\\ &+ \frac{N-i}{N}\Pr(E_{2,n} \mid A_{2,n}C_{i,n})\\ &+ \Pr(E_{0,n} \mid C_{i,n})\Big]\end{aligned} \tag{23c}$$

for the probabilities of one-step transitions between states. Equation 23*a*, for example, states that the probability of moving from the state with i elements conditioned to A_1 to the state with $i + 1$ elements conditioned to A_1 is the product of the probability $(N - i)/N$ that an element not already conditioned to A_1 is sampled and the probability $c\Pr(E_{1,n} \mid A_{2,n}C_{i,n})$ that, under the given circumstances, conditioning occurs.

As defined earlier, we have a Markov process in the conditioning states if the probability of a transition from any state to any other state depends at most on the state existing on the trial preceding the transition. By inspection of Eq. 23 we see that the Markov condition may be satisfied by limiting ourselves to reinforcement schedules in which the probability of a reinforcing event E_i depends at most on the response of the given trial; that is, in learning-theory terminology, to noncontingent and simple contingent schedules. This restriction will be assumed throughout the present section except for a few remarks in which we explicitly consider various lines of generalization.

With these restrictions in mind, we define

$$\pi_{ij} = \Pr(E_{j,n} \mid A_{i,n}),$$

where $j = 0$ to r, $i = 1$ to r, and $\sum_j \pi_{ij} = 1$; that is, the reinforcement on a trial depends at most on the response of the given trial. Further, the reinforcement probabilities do not depend on the trial number. We may then rewrite Eq. 23 as follows:

$$q_{i,i+1} = c\frac{N-i}{N}\pi_{21} \tag{24a}$$

$$q_{i,i} = 1 - c\frac{N-i}{N}\pi_{21} - c\frac{i}{N}\pi_{12} \tag{24b}$$

$$q_{i,i-1} = c\frac{i}{N}\pi_{12}. \tag{24c}$$

Note that we use the notation q_{ij} in place of $\Pr(C_{j,n+1} \mid C_{i,n})$. The reason is that the transition probabilities do not depend on n, given the restrictions on the reinforcement schedule stated above, and the simpler notation expresses this fact.

RESPONSE PROBABILITIES AND MOMENTS. By Axioms S1, S2, and R1 we know that the relation between response probability and the conditioning state is simply

$$\Pr(A_{1,n} \mid C_{i,n}) = \frac{i}{N}.$$

Hence

$$\begin{aligned}\Pr(A_{1,n}) &= \sum_{i=0}^{N} \Pr(A_{1,n} \mid C_{i,n}) \Pr(C_{i,n}) \\ &= \sum_{i=0}^{N} \frac{i}{N} \Pr(C_{i,n}).\end{aligned} \tag{25}$$

But note that by definition of the transition probabilities q_{ij}

$$\begin{aligned}\Pr(C_{i,n}) &= \Pr(C_{0,n-1})q_{0i} + \Pr(C_{1,n-1})q_{1i} + \dots + \Pr(C_{N,n-1})q_{Ni} \\ &= \sum_{j=0}^{N} \Pr(C_{j,n-1})q_{ji}.\end{aligned} \tag{26}$$

The latter expression, together with Eq. 25, serves as the basis for a general recursion in $\Pr(A_{1,n})$:

$$\Pr(A_{1,n}) = \sum_{i=0}^{N} \frac{i}{N} \sum_{j=0}^{N} \Pr(C_{j,n-1})q_{ji}.$$

Now substituting for q_{ji} in terms of Eq. 24 and rearranging the sum we have

$$\begin{aligned}\Pr(A_{1,n}) = \sum_{i=0}^{N} \frac{i}{N} \Pr(C_{i,n-1}) &- c\pi_{12}\sum_{i=1}^{N} \frac{i^2}{N^2} \Pr(C_{i,n-1}) \\ &- c\pi_{21}\sum_{i=0}^{N-1} \frac{i(N-i)}{N^2} \Pr(C_{i,n-1}) \\ &+ c\pi_{21}\sum_{i=0}^{N-1} \frac{(i+1)(N-i)}{N^2} \Pr(C_{i,n-1}) \\ &+ c\pi_{12}\sum_{i=1}^{N} \frac{i(i-1)}{N^2} \Pr(C_{i,n-1}).\end{aligned}$$

The first sum is, by Eq. 25, $\Pr(A_{1,n-1})$. Let us define

$$\alpha_{2,n} = \sum_{i=0}^{N} (i^2/N^2) \Pr(C_{i,n});$$

then the second sum is simply $-c\pi_{12}\alpha_{2,n-1}$. Similarly, the third sum is

$$-c\pi_{21}[\Pr(A_{1,n-1}) - \Pr(C_{N,n-1}) - \alpha_{2,n-1} + \Pr(C_{N,n-1})] = -c\pi_{21}[\Pr(A_{1,n-1}) - \alpha_{2,n-1}],$$

and so forth. Carrying out the summation and simplifying, we obtain the following recursion in $\Pr(A_{1,n})$:

$$\Pr(A_{1,n}) = \left[1 - \frac{c}{N}(\pi_{12} + \pi_{21})\right] \Pr(A_{1,n-1}) + \frac{c}{N}\pi_{21}. \tag{27}$$

This difference equation has the well-known solution (cf. Bush & Mosteller, 1955; Estes, 1959b; Estes & Suppes, 1959)

$$\Pr(A_{1,n}) = \Pr(A_{1,\infty}) - [\Pr(A_{1,\infty}) - \Pr(A_{1,1})]\left[1 - \frac{c}{N}(\pi_{12} + \pi_{21})\right]^{n-1}, \tag{28}$$

where

$$\Pr(A_{1,\infty}) = \frac{\pi_{21}}{\pi_{21} + \pi_{12}}.$$

At this point it will also be instructive to calculate the variance of the distribution of response probabilities $\Pr(A_{1,n} \mid C_{i,n})$. The second raw moment, as defined above, is

$$\alpha_{2,n} = \sum_{i=0}^{N} \frac{i^2}{N^2} \Pr(C_{i,n}) = \sum_{i=0}^{N} \frac{i^2}{N^2} \sum_{j=0}^{N} \Pr(C_{j,n-1}) q_{ji}. \tag{29}$$

Carrying out the summation, as in the case of Eq. 27, we obtain

$$\alpha_{2,n} = \alpha_{2,n-1}\left[1 - \frac{2c}{N}(\pi_{12} + \pi_{21})\right] + \Pr(A_{1,n-1})\left[c\pi_{12}\frac{1}{N^2} + c\pi_{21}\left(\frac{2}{N} - \frac{1}{N^2}\right)\right] + \frac{c}{N^2}\pi_{21}.$$

Subtracting the square of $\Pr(A_{1,n})$, as given in Eq. 28, from $\alpha_{2,n}$ yields the variance of the response probabilities. The second and higher moments of the response probabilities are of experimental interest primarily because they enter into predictions concerning various sequential statistics. We shall return to this point later.

ASYMPTOTIC DISTRIBUTIONS. The pattern model has one particularly advantageous feature not shared by many other learning models that have appeared in the literature. This feature is a simple calculational procedure for generating the complete asymptotic distribution of conditioning states and therefore the asymptotic distribution of responses. The derivation to be given assumes that all elements $q_{i,i-1}$, $q_{i,i}$, $q_{i,i+1}$ of the transition matrix are nonzero; the same technique can be applied if there are zero entries, except, of course, that in forming ratios one must keep the zeros out of the denominators.

As in Sec. 1.3, we let $\lim_{n\to\infty} \Pr(C_{i,n}) = u_i$. The theorem to be proved is that all of the asymptotic conditioning state probabilities u_i can be expressed recursively in terms of u_0; since the u_i's must sum to unity, this recursion suffices to determine the entire distribution.

By Eq. 26 we note that

$$u_0 = u_0 q_{00} + u_1 q_{10},$$

hence

$$\frac{u_0}{u_1} = \frac{q_{10}}{1 - q_{00}} = \frac{q_{10}}{q_{01}}.$$

We now prove by induction that a similar relation holds for any adjacent pair of states; that is,

$$\frac{u_i}{u_{i+1}} = \frac{q_{i+1,i}}{q_{i,i+1}}.$$

For any state i we have by Eq. 26

$$u_i = u_{i-1} q_{i-1,i} + u_i q_{i,i} + u_{i+1} q_{i+1,i}.$$

Rearranging,

$$u_i(1 - q_{i,i}) = u_{i-1} q_{i-1,i} + u_{i+1} q_{i+1,i}.$$

However, under the inductive hypothesis we may replace u_{i-1} by its equivalent $u_i q_{i,i-1}/q_{i-1,i}$. Hence

$$\begin{aligned} u_i(1 - q_{i,i}) &= \frac{u_i q_{i,i-1} q_{i-1,i}}{q_{i-1,i}} + u_{i+1} q_{i+1,i} \\ &= u_i q_{i,i-1} + u_{i+1} q_{i+1,i} \end{aligned}$$

or

$$u_i(1 - q_{i,i} - q_{i,i-1}) = u_{i+1} q_{i+1,i}.$$

However, $1 - q_{i,i} - q_{i,i-1} = q_{i,i+1}$, since $q_{i,i-1} + q_{i,i} + q_{i,i+1} = 1$, and therefore

$$\frac{u_i}{u_{i+1}} = \frac{q_{i+1,i}}{q_{i,i+1}},$$

which concludes the proof.

Thus we may write

$$u_1 = \frac{q_{01}}{q_{10}} u_0, \qquad u_2 = \frac{q_{12}}{q_{21}} u_1 = \frac{q_{12} q_{01}}{q_{21} q_{10}} u_0,$$

and so forth. Since the u_i's must sum to unity, u_0 also is determined. To illustrate the application of this technique, we consider some simple cases. For the noncontingent case discussed in Sec. 1.3.

$$\pi = \pi_{21} = \pi_{11}$$
$$1 - \pi = \pi_{12} = \pi_{22}.$$

By Eq. 24 we have

$$q_{i,i+1} = c\,\frac{N-i}{N}\,\pi$$

$$q_{i,i-1} = c\,\frac{i}{N}\,(1-\pi).$$

Applying the technique of the previous paragraph,

$$\frac{u_1}{u_0} = \frac{c\pi}{c(1/N)(1-\pi)} = \frac{N\pi}{(1-\pi)}$$

$$\frac{u_2}{u_1} = \frac{\pi c[(N-1)/N]}{(1-\pi)c(2/N)} = \frac{(N-1)\pi}{2(1-\pi)}$$

and in general

$$\frac{u_k}{u_{k-1}} = \frac{(N-k+1)\pi}{k(1-\pi)}.$$

This result has two interesting features. First, we note that the asymptotic probabilities are independent of the conditioning parameter c. Second, the ratio of u_k to u_{k-1} is the same as that of neighboring terms

$$\binom{N}{k}\pi^k(1-\pi)^{N-k} \quad \text{and} \quad \binom{N}{k-1}\pi^{k-1}(1-\pi)^{N-k+1}$$

in the expansion of $[\pi + (1-\pi)]^N$. Therefore the asymptotic probabilities in this case are binomially distributed. For a population of subjects whose learning is described by the model, the limiting proportion of subjects having all N patterns conditioned to A_1 is π^N; the proportion having all but one of the N patterns conditioned to A_1 is $N\pi^{N-1}(1-\pi)$; and so on.

For the case of simple contingent reinforcement,

$$\frac{u_k}{u_{k-1}} = \frac{(N-k+1)\pi_{21}c}{N}\Big/\frac{k\pi_{12}c}{N} = \frac{(N-k+1)\pi_{21}}{k\pi_{12}}.$$

Again we note that the u_i are independent of c. Further the ratio u_k to u_{k-1} is the same as that of

$$\binom{N}{k}\pi_{21}^k\pi_{12}^{N-k} \quad \text{to} \quad \binom{N}{k-1}\pi_{21}^{k-1}\pi_{12}^{N-k+1}.$$

Therefore the asymptotic state probabilities are the terms in the expansion of

$$\left(\frac{\pi_{21}}{\pi_{21}+\pi_{12}}+\frac{\pi_{12}}{\pi_{21}+\pi_{12}}\right)^N.$$

Explicit formulas for state probabilities are useful primarily as intermediary expressions in the derivation of other quantities. In the special case of the pattern model (unlike other types of stimulus sampling models) the strict determination of the response on any trial by the conditioning state of the trial sample permits a relatively direct empirical interpretation, for the moments of the distribution of state probabilities are identical with the moments of the response random variable. Thus in the simple contingent case we have immediately for the mean and variance of the response random variable $\mathbf{A}_\infty$

$$E(\mathbf{A}_\infty)=\sum_{k=1}^{N}\frac{k}{N}\binom{N}{k}\left(\frac{\pi_{21}}{\pi_{21}+\pi_{12}}\right)^k\left(\frac{\pi_{12}}{\pi_{21}+\pi_{12}}\right)^{N-k}=\frac{\pi_{21}}{\pi_{21}+\pi_{12}}$$

and

$$\operatorname{Var}(\mathbf{A}_\infty)=\sum_{k=1}^{N}\frac{k^2}{N^2}\binom{N}{k}\left(\frac{\pi_{21}}{\pi_{21}+\pi_{12}}\right)^k\left(\frac{\pi_{12}}{\pi_{21}+\pi_{12}}\right)^{N-k}-[E(\mathbf{A}_\infty)]^2$$

$$=\frac{\pi_{21}\pi_{12}}{(\pi_{21}+\pi_{12})^2}.$$

A bit of caution is needed in applying this last expression to data. If we select some fixed trial n (large enough so that the learning process may be assumed asymptotic), then the theoretical variance for the A_1-response totals of a number of independent samples of K subjects on trial n is simply $K[\pi_{21}\pi_{12}/(\pi_{21}+\pi_{12})^2]$ by the familiar theorem for the variance of a sum of independent random variables. However, this expression does not hold for the variance of A_1-response totals over a block of K successive trials. The additional considerations involved in the latter case are discussed in the next section.

2.2 Treatment of the Simple Noncontingent Case

In this section we shall consider various predictions that may be derived from the pattern model for simple predictive behavior in a two-choice situation with noncontingent reinforcement. Each trial in the reference experiment begins with the presentation of a ready signal; the subject's task is to respond to the signal by operating one of a pair of response keys, A_1 or A_2, indicating his prediction as to which of two reinforcing lights will appear. The reinforcing lights are programmed by the experimenter to occur in random sequence, exactly one on each trial, with probabilities that are constant throughout the series and independent of the subject's behavior.

For illustrative purposes, we shall use data from two experiments of this sort. In one of these, henceforth designated the 0.6 series, 30 subjects were run, each for a series of 240 trials, with probabilities of 0.6 and 0.4 for the two reinforcing lights. Details of the experimental procedure, and a more complete analysis of the data than we shall undertake here, are given in Suppes & Atkinson (1960, Chapter 10). In the other experiment, henceforth designated the 0.8 series, 80 subjects were run, each for a series of 288 trials, with probabilities of 0.8 and 0.2 for the two reinforcing lights. Details of the procedure and results have been reported by Friedman et al. (1960). A possibly important difference between the conditions of the two experiments is that in the 0.6 series the subjects were new to this type of experiment, whereas in the 0.8 series the subjects were highly practiced, having had experience with a variety of noncontingent schedules in two previous experimental sessions.

For our present purposes it will suffice to consider only the simplest possible interpretation of the experimental situation in terms of the pattern model. Let O_1 denote the more frequently occurring reinforcing light and O_2 the less frequent light. We then postulate a one-to-one correspondence between the appearance of light O_i and the reinforcing event E_i which is associated with A_i (the response of predicting O_i). Also we assume that the experimental conditions determine a set of N distinct stimulus patterns, exactly one of which is present at the onset of any given trial. Since, in experiments of the sort under consideration, the experimenter usually presents the same ready signal at the beginning of every trial, we might assume that N would necessarily equal unity. However, we shall not impose this restriction on the model. Rather, we shall let N appear as a free parameter in theoretical expressions; then we shall seek to determine from the data the value of N required to minimize the disparities between theoretical and observed values.

If the data of a particular experiment yield an estimate of N greater than unity and if, with this estimate, the model provides a satisfactory account of the empirical relationships in question, we shall conclude that the learning process proceeds as described by the model but that, regardless of the experimenter's intention, the subjects are sampling a population of stimulus patterns. The pattern effective at the onset of a given trial might comprise the experimenter's ready signal together with stimulus traces (perhaps verbally mediated) of the reinforcing events and responses of one or more preceding trials.

It will be apparent that the pattern model could scarcely be expected to provide a completely adequate account of the data of two-choice experiments run under the conditions sketched above. First, if the stimulus patterns to which the subject responds include cues from preceding events, then it is extremely unlikely that all of the available patterns would have equal sampling probabilities as assumed in the model. Second, the different patterns must have component cues in common, and these would be expected to yield transfer effects (at least on early trials) so that the response to a pattern first sampled on trial n would be influenced by conditioning that occurred when components of that pattern were present on earlier trials. However, the pattern model assumes that all of the patterns available for sampling are distinct in the sense that reinforcement of a response to one pattern has no effect on response probabilities associated with other patterns.

Despite these complications, many investigators (e.g., Suppes & Atkinson, 1960; Estes, 1961b; Suppes & Ginsberg, 1962; Bower, 1961) have found it a useful strategy to apply the pattern model in the simple form presented in the preceding section. The goal in these applications is not the perhaps impossible one of accounting for every detail of the experimental results but rather the more modest, yet realizable, one of obtaining valuable information about various theoretical assumptions by comparing manageably simple models that embody different combinations of assumptions. This procedure is illustrated in the remainder of the section.

SEQUENTIAL PREDICTIONS. We begin our application of the pattern model with a discussion of sequential statistics. It should be emphasized that one of the major contributions of mathematical learning theory has been to provide a framework within which the sequential aspects of learning can be scrutinized. Before the development of mathematical models little attention was paid to trial-by-trial phenomena; at the present time, for many experimental problems, such phenomena are viewed as the most interesting aspect of the data.

Although we consider only the noncontingent case, the same methods may be used to obtain results for more general reinforcement schedules. We shall develop the proofs in terms of two responses, but the results hold for any number of alternatives. If there are r responses in a given experimental application, any one response can be denoted A_1 and the rest regarded as members of a single class, A_2.

We consider first the probability of an A_1 response, given that it occurred and was reinforced on the preceding trial; that is, $\Pr(A_{1,n+1} \mid E_{1,n}A_{1,n})$. It is convenient to deal first with the joint probability $\Pr(A_{1,n+1}E_{1,n}A_{1,n})$ and to conditionalize later. First we note that

$$\Pr(A_{1,n+1}E_{1,n}A_{1,n}) = \sum_{i,j} \Pr(A_{1,n+1}C_{j,n+1}E_{1,n}A_{1,n}C_{i,n}), \tag{30}$$

and that $\Pr(A_{1,n+1}C_{j,n+1}E_{1,n}A_{1,n}C_{i,n})$ may be expressed in terms of conditional probabilities as

$$\Pr(A_{1,n+1} \mid C_{j,n+1}E_{1,n}A_{1,n}C_{i,n}) \Pr(C_{j,n+1} \mid E_{1,n}A_{1,n}C_{i,n}) \cdot \Pr(E_{1,n} \mid A_{1,n}C_{i,n}) \Pr(A_{1,n} \mid C_{i,n}) \Pr(C_{i,n}).$$

But from the sampling and response axioms the probability of a response on trial n is determined solely by the conditioning state on trial n; that is, the first factor in the expansion can be rewritten simply as $\Pr(A_{1,n+1} \mid C_{j,n+1})$. Further, by Axiom R1, we have

$$\Pr(A_{1,n+1} \mid C_{j,n+1}) = j/N.$$

For the noncontingent case the probability of an E_1 on any trial is independent of previous events and consequently we may write

$$\Pr(E_{1,n} \mid A_{1,n}C_{i,n}) = \pi.$$

Next, we note that

$$\Pr(C_{j,n+1} \mid E_{1,n}A_{1,n}C_{i,n}) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j; \end{cases}$$

that is, an element conditioned to A_1 is sampled on trial n (since an A_1-response occurs on n) and thus by Axiom C2 no change in the conditioning state can occur.

Putting these results together and substituting in Eq. 30, we obtain

$$\begin{aligned} \Pr(A_{1,n+1}E_{1,n}A_{1,n}) &= \pi \sum_i \frac{i^2}{N^2} \Pr(C_{i,n+1} \mid E_{1,n}A_{1,n}C_{i,n}) \Pr(C_{i,n}) \\ &= \pi \sum_i \frac{i^2}{N^2} \Pr(C_{i,n}) \\ &= \pi\alpha_{2,n}, \end{aligned} \tag{31a}$$

and

$$\begin{aligned} \Pr(A_{1,n+1} \mid E_{1,n}A_{1,n}) &= \frac{\pi\alpha_{2,n}}{\Pr(E_{1,n}A_{1,n})} \\ &= \frac{\alpha_{2,n}}{\Pr(A_{1,n})}. \end{aligned} \tag{31b}$$

In order to express this conditional probability in terms of the parameters π, c, N, and $\Pr(A_{1,1})$, we simply substitute into Eq. 31*b* the expression given for $\Pr(A_{1,n})$ in Eq. 28 and the corresponding expression for $\alpha_{2,n}$ that would be given by the solution of the difference equation (Eq. 29). Unfortunately, the expression so obtained is extremely cumbersome to work with. Consequently it is usually preferable in working with data to proceed in a different way.

Suppose the data to be treated consist of proportions of occurrences of the various trigrams $A_{k,n+1}E_{j,n}A_{i,n}$ over blocks of M trials. If, for example, $M = 5$, then in the protocol

Trial	1	2	3	4	5
Event	A_1E_1	A_1E_1	A_2E_1	A_1E_1	A_1E_2

There are four opportunities for such trigrams. The combination $A_{1,n+1} \cdot E_{1,n}A_{1,n}$ occurs on two of these, $A_{2,n+1}E_{1,n}A_{1,n}$ on one and $A_{1,n+1}E_{1,n}A_{2,n}$ on the other; hence the proportions of occurrence of these trigrams are 0.50, 0.25, and 0.25, respectively. To deal theoretically with quantities such as these, we need only average both sides of Eq. 31*a* (and the corresponding expressions for other trigrams) over the appropriate block of trials, obtaining, for example, for the block running from trial n through trial $n + M - 1$

$$p_{111} = \frac{1}{M}\sum_{n'=n}^{n+M-1} \Pr(A_{1,n'+1}E_{1,n'}A_{1,n'}) = \frac{\pi}{M}\sum_{n'=n}^{n+M-1} \alpha_{2,n'} = \pi\bar{\alpha}_2(n, M), \quad (32a)$$

where $\bar{\alpha}_2(n, M)$ is the average value of the second moment of the response probabilities over the given trial block. By strictly analogous methods we can derive theoretical expressions for other trigram proportions:

$$p_{112} = \frac{1}{M}\sum_{n'=n}^{n+M-1} \Pr(A_{1,n'+1}E_{1,n'}A_{2,n'})$$

$$= \pi\left[\left(1 - \frac{c}{N}\right)\bar{\alpha}_1(n, M) + \frac{c}{N} - \bar{\alpha}_2(n, M)\right], \quad (32b)$$

$$p_{121} = \frac{1}{M}\sum_{n'=n}^{n+M-1} \Pr(A_{1,n'+1}E_{2,n'}A_{1,n'})$$

$$= (1 - \pi)\left[\bar{\alpha}_2(n, M) - \frac{c}{N}\bar{\alpha}_1(n, M)\right], \quad (32c)$$

$$p_{122} = \frac{1}{N}\sum_{n'=n}^{n+M-1} \Pr(A_{1,n'+1}E_{2,n'}A_{2,n'})$$

$$= (1 - \pi)[\bar{\alpha}_1(n, M) - \bar{\alpha}_2(n, M)], \quad (32d)$$

and so on; the quantity $\bar{\alpha}_1(n, M)$ denoting the average A_1-probability (or, equivalently, the proportion of A_1-responses) over the given trial block.

Now the average moments $\bar{\alpha}_i$ can be treated as parameters to be estimated from the data in order to mediate theoretical predictions. To illustrate, let us consider a sample of data from the 0.8 series. Over the first 12 trials of the $\pi = 0.8$ series, the observed proportion of A_1-responses for the group of 80 subjects was 0.63 and the observed values for the trigrams of Eq. 32*a*-*d* were $p_{111} = 0.379$, $p_{112} = 0.168$, $p_{121} = 0.061$, and $p_{122} = 0.035$. Using p_{111} to estimate $\bar{\alpha}_2(1, 12)$, we have from Eq. 32*a*

$$0.379 = 0.8[\bar{\alpha}_2(1, 12)],$$

which yields as our estimate

$$\hat{\bar{\alpha}}_2(1, 12) = 0.47.$$

Now we are in a position to predict the value of p_{122}. Substituting the appropriate parameter values into Eq. 32*d*, we have

$$p_{122} = 0.2(0.63 - 0.47) = 0.032,$$

which is not far from the observed value of 0.035. Proceeding similarly, we can use Eq. 32*b* to estimate c/N, namely,

$$p_{112} = 0.168 = 0.8\left[\left(1 - \frac{c}{N}\right)(0.63) + \frac{c}{N} - 0.47\right],$$

from which

$$\frac{\hat{c}}{N} = 0.135.$$

With this estimate in hand, together with those already obtained for the first and second moments, we can substitute into Eq. 32*c* and predict the value of p_{121}:

$$\begin{aligned} p_{121} &= 0.2[0.47 - 0.135(0.63)] \\ &= 0.077, \end{aligned}$$

which is somewhat high in relation to the observed value of 0.061.

It should be mentioned that the simple estimation method used above for illustrative purposes would be replaced, in a serious application of the model, by a more systematic procedure. For example, one might simultaneously estimate $\bar{\alpha}_2$ and c/N by least squares, employing all eight of the p_{ijk}; this procedure would yield a better over-all fit of the theoretical and observed values.

A limitation of the method just described is that it permits estimation of the ratio c/N but not estimation of c and N separately. Fortunately, in the asymptotic case, the expressions for the moments α_i are simple enough so that expressions for the trigrams in terms of the parameters are manageable; and it turns out to be easy to evaluate the conditioning parameter and the number of elements from these expressions. The limit of $\alpha_{1,n}$ for large n is, of course, π in the simple noncontingent case. The limit, α_2, of $\alpha_{2,n}$ may be obtained from the solution of Eq. 29; however, a simpler method of obtaining the same result is to note that, by definition,

$$\alpha_2 = \sum_i \frac{i^2}{N^2} u_i,$$

where u_i again represents the asymptotic probability of the state in which i elements are conditioned to A_1. Recalling that the u_i are terms of the binomial distribution, we may then write

$$\alpha_2 = \sum_i \frac{i^2}{N^2}\binom{N}{i}\pi^i(1-\pi)^{N-i}$$
$$= \frac{1}{N^2}\sum_i i^2\binom{N}{i}\pi^i(1-\pi)^{N-i}.$$

The summation is the second raw moment of the binomial distribution with parameter π and sample size N. Therefore

$$\alpha_2 = \frac{N\pi(1-\pi) + N^2\pi^2}{N^2}$$
$$= \frac{\pi(1-\pi)}{N} + \pi^2. \tag{33}$$

Using Eq. 33 and the fact that $\lim \Pr(A_{1,n}) = \pi$, we have

$$\lim_{n\to\infty} \Pr(A_{1,n+1} \mid E_{1,n}A_{1,n}) = \pi\left(1 - \frac{1}{N}\right) + \frac{1}{N}. \tag{34a}$$

By identical methods we can establish that

$$\lim \Pr(A_{1,n+1} \mid E_{1,n}A_{2,n}) = \pi\left(1 - \frac{1}{N}\right) + \frac{c}{N}, \tag{34b}$$

$$\lim \Pr(A_{1,n+1} \mid E_{2,n}A_{1,n}) = \pi\left(1 - \frac{1}{N}\right) + \frac{1-c}{N}, \tag{34c}$$

and

$$\lim \Pr(A_{1,n+1} \mid E_{2,n}A_{2,n}) = \pi\left(1 - \frac{1}{N}\right). \tag{34d}$$

With these formulas in hand, we need only apply elementary probability theory to obtain expressions for dependencies of responses on responses or responses on reinforcements, namely,

$$\lim \Pr(A_{1,n+1} \mid A_{1,n}) = \pi + \frac{(1-c)(1-\pi)}{N} \tag{35a}$$

$$\lim \Pr(A_{1,n+1} \mid A_{2,n}) = \pi - \frac{(1-c)\pi}{N} \tag{35b}$$

$$\lim \Pr(A_{1,n+1} \mid E_{1,n}) = \left(1 - \frac{c}{N}\right)\pi + \frac{c}{N} \tag{35c}$$

$$\lim \Pr(A_{1,n+1} \mid E_{2,n}) = \left(1 - \frac{c}{N}\right)\pi. \tag{35d}$$

Given a set of trigram proportions from the asymptotic data of a two-choice experiment, we are now in a position to achieve a test of the model by using part of the data to estimate the parameters c and N, and then substituting these estimates into Eq. 34*a-d* and 35*a-d* to predict the values of all eight of these sequential statistics. We shall illustrate this procedure with the data of the 0.6 series. The observed transition frequencies $F(A_{i,n+1} \mid E_{j,n}A_{k,n})$ for the last 100 trials, aggregated over subjects, are as follows:

	A_1	A_2
A_1E_1	748	298
A_1E_2	394	342
A_2E_1	462	306
A_2E_2	186	264

An estimate of the asymptotic probability of an A_1-response given an A_1E_1-event on the preceding trial can be obtained by dividing the first entry in row one by the sum of the row; that is, $\Pr(A_1 \mid E_1A_1) = 748/(748 + 298) = 0.715$. But, if we turn to Eq. 34*a*, we note that $\lim \Pr(A_{1,n+1} \mid E_{1,n}A_{1,n}) = \pi(1 - 1/N) + 1/N$. Hence, letting $0.715 = 0.6(1 - 1/N) + 1/N$, we obtain an estimate[7] of $N = 3.48$. Similarly $\Pr(A_1 \mid E_1A_2) = 462/(462 + 306) = 0.602$, which by Eq. 34b is an estimate of $\pi(1 - 1/N) + c/N$; using our values of π and N we find that $c/N = 0.174$ and $c = 0.605$.

Having estimated c and N, we may now generate predictions for any of our asymptotic quantities. Table 3 presents predicted and observed values for the quantities given in Eq. 34*a* to Eq. 35*d*. Considering that only two degrees of freedom have been utilized in estimating parameters, the close correspondence between theoretical and observed quantities in Table 3 may be interpreted as giving considerable support to the assumptions of the model. A similar analysis of the asymptotic data from the 0.8 series, which has been reported elsewhere (Estes, 1961b), yields comparable agreement between theoretical and observed trigram proportions. The estimate of c/N for the 0.8 data is very close to that for the 0.6 data (0.172 versus 0.174), but the estimates of c and N (0.31 and 1.84, respectively) are both smaller for the 0.8 data. It appears that the more highly practiced subjects of the 0.8 series are, on the average, sampling from a smaller population of stimulus patterns and at the same time are less responsive to the reinforcing lights than the more naïve subjects of the 0.6 series.

[7] For any one subject, N must, of course, be an integer. The fact that our estimation procedures generally yield nonintegral values for N may signify that N varies somewhat between subjects, or it may simply reflect some contamination of the data by sources of experimental error not represented in the model.

Since no model can be expected to give a perfect account of fallible data arising from real experiments (as distinguished from the idealized experiments to which the model should apply strictly), it is difficult to know how to evaluate the goodness-of-fit of theoretical to observed values. In practice, investigators usually proceed on a largely intuitive basis, evaluating the fit in a given instance against that which it appears reasonable to hope for in the light of what is known about the precision of experimental control and measurement. Statistical tests of goodness-of-fit are sometimes

Table 3 Predicted (Pattern Model) and Observed Values of Sequential Statistics for Final 100 Trials of the 0.6 Series

Asymptotic Quantity	Predicted	Observed
$\Pr(A_1 \mid E_1A_1)$	0.715	0.715
$\Pr(A_1 \mid E_2A_1)$	0.541	0.535
$\Pr(A_1 \mid E_1A_2)$	0.601	0.601
$\Pr(A_1 \mid E_2A_2)$	0.428	0.413
$\Pr(A_1 \mid A_1)$	0.645	0.641
$\Pr(A_1 \mid A_2)$	0.532	0.532
$\Pr(A_1 \mid E_1)$	0.669	0.667
$\Pr(A_1 \mid E_2)$	0.496	0.489

possible (discussions of some tests which may be used in conjunction with stimulus sampling models are given in Suppes & Atkinson, 1960); however, statistical tests are not entirely satisfactory, taken by themselves, for a sufficiently precise test will often indicate significant differences between theoretical and observed values even in cases in which the agreement is as close as could reasonably be hoped for. Generally, once a degree of descriptive accuracy that appears satisfactory to investigators familiar with the given area has been attained, further progress must come largely via differential tests of alternative models.

In the case of the two-choice noncontingent situation the ingredients for one such test are immediately at hand; for we developed in Sec. 1.3 a one-element, guessing-state model that is comparable to the N-element model with respect to the number of free parameters and that to many might seem equally plausible on psychological grounds. These models embody the all-or-none assumption concerning the formation of learned associations, but they differ in the means by which they escape the deterministic features of the simple one-element model. It will be recalled that the one-element model cannot handle the sequential statistics considered in this section because it requires, for example, a probability of unity for response A_i on any trial following a trial on which A_i occurred and was reinforced. In the N-element model (with $N \geqslant 2$), there is no such constraint, for the stimulus pattern present on the preceding reinforced trial may be replaced by another pattern, possibly conditioned to a different response, on the following trial. In the guessing-state model there is no strict determinacy, since the A_i-response may occur on the reinforced trial by guessing if the subject is in state C_0; and, if the reinforcement were not effective, a different response might occur, again through guessing, on the following trial.

The case of the guessing-state model with $c = 0$ (c, it will be recalled, being the counterconditioning parameter) provides a two-parameter model which may be compared with the two-parameter, N-element model. We will require an expression for at least one of the trigram proportions studied in connection with the N-element model. Let us take $\Pr(A_{1,n+1}E_{1,n}A_{1,n})$ for this purpose. In Sec. 1.3 we obtained an expression for $\Pr(A_{1,n+1} \mid E_{1,n}A_{1,n})$ for the case in which $c = 0$, and thus we can write at once

$$\Pr(A_{1,n+1}E_{1,n}A_{1,n}) = \pi\{u_{1,n} + \tfrac{1}{2}u_{0,n}[c'' + (1 - c'')\tfrac{1}{2}]\}. \qquad (36a)$$

Since we are interested only in the asymptotic case, we drop the n-subscript from the right-hand side of Eq. 36a and have for the desired theoretical asymptotic expression

$$p_{111} = \pi[u_1 + u_0(1 + c'')\tfrac{1}{4}]. \qquad (36b)$$

Substituting now into Eq. 36b the expressions for u_1 and u_0 derived in Sec. 1.3, we obtain finally

$$p_{111} = \pi^2 \frac{[4\pi + (1 - \pi)\epsilon(1 - c'')]}{4[\pi^2 + (1 - \pi)^2 + \pi(1 - \pi)\epsilon]}. \qquad (36c)$$

To apply this model to the asymptotic data of the 0.6 series, we may first evaluate the parameter ϵ by setting the observed proportion of A_1-responses over the terminal 100 trials, 0.593, equal to the right-hand side of Eq. 21 and solving for ϵ, namely,

$$0.593 = \frac{\pi[\pi + (1 - \pi)(\epsilon/2)]}{\pi^2 + (1 - \pi)^2 + \pi(1 - \pi)\epsilon}$$

$$= \frac{0.6(0.6 + 0.2\epsilon)}{0.52 + 0.24\epsilon},$$

and

$$\epsilon = 2.315.$$

Now, by introducing this value for ϵ into Eq. 36c and simplifying, we obtain the prediction

$$p_{111} = 0.2782 + 0.0775c''.$$

Since the observed value of p_{111} for the 0.6 data is 0.249, it is apparent that no matter what value (in the admissible range $0 < c'' \leqslant 1$) is chosen for the parameter c'' the value predicted from the guessing state model will be too large. Further analysis, using the methods illustrated, makes it clear that for no combination of parameter estimates can the guessing-state model achieve predictive accuracy comparable to that demonstrated for the N-element model in Table 3. Although this one comparison cannot be considered decisive, we might be inclined to suspect that for interpretation of two-choice, probability learning the notion of a reaccessible guessing state is on the wrong track, whereas the N-element sampling model merits further investigation.

MEAN AND VARIANCE OF A_1 RESPONSE PROPORTION. By letting $\pi_{11} = \pi_{21} = \pi$ in Eq. 28, we have immediately an expression for the probability of an A_1-response on trial n in the noncontingent case, namely,

$$\Pr(A_{1,n}) = \pi - [\pi - \Pr(A_{1,1})]\left(1 - \frac{c}{N}\right)^{n-1}. \tag{37}$$

If we define a response random variable $\mathbf{A}_n$ which equals 1 or 0 as A_1 or A_2, respectively, occurs on trial n, then the right side of Eq. 37 also represents the expectation of this random variable on trial n. The expected number of A_1-responses in a series of K trials is then given by the summation of Eq. 37 over trials,

$$E(\bar{\mathbf{A}}_K) = \sum_{n=1}^{K} E(\mathbf{A}_n) = K\pi - \frac{N}{c}[\pi - \Pr(A_{1,1})]\left[1 - \left(1 - \frac{c}{N}\right)^K\right]. \tag{38}$$

In experimental applications we are frequently interested in the learning curve obtained by plotting the proportion of A_1-responses per K-trial block. A theoretical expression for this learning function is readily obtained by an extension of the method used to derive Eq. 38. Let x be the ordinal number of a K-trial block running from trial $K(x-1)+1$ to Kx, where $x = 1, 2, \ldots$, and define $P(x)$ as the proportion of A_1-responses in block x. Then

$$\begin{aligned} P(x) &= \frac{1}{K}\left[\sum_{n=1}^{Kx} \Pr(A_{1,n}) - \sum_{n=1}^{K(x-1)} \Pr(A_{1,n})\right] \\ &= \pi - \frac{N}{Kc}[\pi - \Pr(A_{1,1})]\left[1 - \left(1 - \frac{c}{N}\right)^K\right]\left(1 - \frac{c}{N}\right)^{K(x-1)}. \end{aligned} \tag{39a}$$

The value of $\Pr(A_{1,1})$ should be in the neighborhood of 0.5 if response bias does not exist. However, to allow for sampling deviations we may eliminate $\Pr(A_{1,1})$ in favor of the observed value of $P(1)$. This can be done in the following way. Note that

$$P(1) = \pi - \frac{N}{Kc}[\pi - \Pr(A_{1,1})]\left[1 - \left(1 - \frac{c}{N}\right)^K\right].$$

Solving for $[\pi - \Pr(A_{1,1})]$ and substituting the result in Eq. 39*a*, we obtain

$$P(x) = \pi - [\pi - P(1)]\left(1 - \frac{c}{N}\right)^{K(x-1)}. \tag{39b}$$

Applications of Eq. 39*b* to data have led to results that are satisfying in some respects but perplexing in others (see, e.g., Estes, 1959a). In most instances the implication that the learning curve should have π as an asymptote has been borne out (Estes, 1961b, 1962), and further, with a suitable choice of values for c/N, the curve represented by Eq. 39*b* has served to describe the course of learning. However, in experiments run with naïve subjects, as has been nearly always the case, the value of c/N required to fit the mean learning curve has been substantially smaller than the value required to handle the sequential statistics discussed in Sec. 2.1. Consider, for example, the learning curve for the 0.6 series plotted by 20 trial blocks. The observed value of $P(1)$ is 0.48 and the value of c/N estimated from the sequential statistics of the second 20-trial block is 0.12. With these parameter values, Eq. 39*b* yields a prediction of 0.59 for $P(3)$ and the theoretical curve is essentially at asymptote from block 4 on. The empirical learning curve, however, does not approach 0.59 until block 6 and is still short of asymptote at the end of 12 blocks, the mean proportion of A_1-responses over the last five blocks being 0.593 (Suppes & Atkinson, 1960, p. 197).

In the case of the 0.8 series there is a similar disparity between the value of c/N estimated from the sequential statistics and the value estimated from the mean learning curve. As we have already noted, an optimal account of the trigram proportions $\Pr(A_{k,n+1}E_{j,n}A_{i,n})$ requires a c/N-value of approximately 0.17. But, if this estimate is substituted into Eq. 39*a*, the predicted A_1-frequency in the first block of 12 trials is 0.67, compared to an observed value of 0.63, and the theoretical curve runs appreciably above the empirical curve for another five blocks. A c/N-value of 0.06 yields a satisfactory graduation of the observed mean curve in terms of Eq. 39*a*, and a fit to the trigrams that does not look bad by usual standards for prediction in learning experiments. However, comparing predictions based on the two c/N-estimates for the trigrams that contain this parameter, we see that the estimate of 0.17 is distinctly superior. For the trigrams averaged over the first 12 trials, the result is as follows:

	Observed	Theoretical: $c/N = 0.17$	Theoretical: $c/N = 0.06$
p_{112}	0.168	0.177	0.144
p_{121}	0.061	0.073	0.087
p_{212}	0.121	0.119	0.152
p_{221}	0.062	0.053	0.039

The reason for this discrepancy in the value of c/N required to give optimal descriptions of two different aspects of the data is not clear even after much investigation. One contributing factor might be individual differences in learning rates (c/N-values) among subjects; these would be expected to affect the two types of statistics differently. However, in the case of the 0.8 series, when a more homogeneous subgroup of subjects (the middle 50% on total A_1 frequency) is analyzed, the disparity, although somewhat reduced, is not eliminated; optimal c/N-values for the mean curve and the trigram statistics are now 0.08 and 0.15, respectively. The principal source of the remaining discrepancy in this homogeneous subgroup is a much smaller increment in A_1-frequency from the first to the second 12-trial block than is predicted. Over the first three blocks the observed proportions are 0.633, 0.665, and 0.790; the proportions predicted from Eq. 39*a* with $c/N = 0.15$ run 0.657, 0.779, and 0.800. A possible explanation is that in the early part of the series the subjects are responding to cues, perhaps verbal in character, which are discarded (i.e., are not resampled) when they fail to elicit consistently correct responding. An interpretation of this sort could be incorporated into the model and subjected to formal testing, but this has not yet been done. In any event, we can see that analyses of data in terms of a model enables us to determine precisely which aspects of the subjects' behavior are and which are not accounted for in terms of a particular set of assumptions.

Next to the mean learning curve, the most frequently used behavioral measure in learning experiments is perhaps the variance of response occurrences in a block of trials. Predicting this variance from a theoretical model is an exceedingly taxing assignment; for the effects of individual differences in learning rate, together with those of all sources of experimental error not represented in the model, must be expected to increase the observed response variance. However, this statistic is relatively easy to compute for the pattern model, and the derivation may serve as a prototype for derivations of similar expressions in other learning models. For simplicity, we shall limit consideration here to the case of the variance of A_1-response frequency in a trial block after the mean curve has reached asymptote.

As a preliminary to computation of the variance, we require a statistic that is also of interest in its own right, the covariance of A_1-responses on any two trials; that is,

$$\begin{aligned}\mathrm{Cov}\,(\mathbf{A}_{n+k}\mathbf{A}_n) &= E(\mathbf{A}_{n+k}\mathbf{A}_n) - E(\mathbf{A}_{n+k})\,E(\mathbf{A}_n)\\ &= \Pr(A_{1,n+k}A_{1,n}) - \Pr(A_{1,n+k})\Pr(A_{1,n}).\end{aligned} \tag{40}$$

First, we can establish by induction that

$$\Pr(A_{1,n+k}A_{1,n}) = \pi \Pr(A_{1,n}) - [\pi \Pr(A_{1,n}) - \Pr(A_{1,n+1}A_{1,n})]\left(1 - \frac{c}{N}\right)^{k-1}.$$

This formula is obviously an identity for $k = 1$. Thus, assuming that the formula holds for trials n and $n + k$, we may proceed to establish it for trials n and $n + k + 1$. First we use our standard procedure to expand the desired quantity in terms of reinforcing events and states of conditioning. Letting $C_{j,n}$ denote the state in which exactly j of the N elements are conditioned to response A_1, we may write

$$\begin{aligned}\Pr(A_{1,n+k+1}A_{1,n}) &= \sum_{i,j} \Pr(A_{1,n+k+1}E_{i,n+k}C_{j,n+k}A_{1,n})\\ &= \sum_{i,j} \Pr(A_{1,n+k+1} \mid E_{i,n+k}C_{j,n+k}A_{1,n}) \Pr(E_{i,n+k}C_{j,n+k}A_{1,n}).\end{aligned}$$

Now we can make use of the assumptions that specify the noncontingent case to simplify the second factor to

$$\pi \Pr(C_{j,n+k}A_{1,n}) \quad \text{and} \quad (1 - \pi)\Pr(C_{j,n+k}A_{1,n})$$

for $i = 1, 2$, respectively. Also, we may apply the learning axioms to the first factor to obtain

$$\begin{aligned}\Pr(A_{1,n+k+1} \mid E_{1,n+k}C_{j,n+k}A_{1,n}) &= \frac{j^2}{N^2} + \left(1 - \frac{j}{N}\right)\left[\frac{(1-c)j}{N} + \frac{c(j+1)}{N}\right]\\ &= \left(1 - \frac{c}{N}\right)\frac{j}{N} + \frac{c}{N}\end{aligned}$$

and

$$\Pr(A_{1,n+k+1} \mid E_{2,n+k}C_{j,n+k}A_{1,n}) = \left(1 - \frac{c}{N}\right)\frac{j}{N}.$$

Combining these results, we have

$$\begin{aligned}&\Pr(A_{1,n+k+1}A_{1,n})\\ &\quad= \sum_j \left\{\pi\left[\left(1 - \frac{c}{N}\right)\frac{j}{N} + \frac{c}{N}\right] + (1-\pi)\left(1 - \frac{c}{N}\right)\frac{j}{N}\right\}\Pr(C_{j,n+k}A_{1,n})\\ &\quad= \sum_j \left\{\left(1 - \frac{c}{N}\right)\frac{j}{N} + \pi\frac{c}{N}\right\}\Pr(C_{j,n+k}A_{1,n})\\ &\quad= \left(1 - \frac{c}{N}\right)\Pr(A_{1,n+k}A_{1,n}) + \pi\frac{c}{N}\Pr(A_{1,n}).\end{aligned}$$

Substitution into this expression in terms of our inductive hypothesis yields

$$\Pr(A_{1,n+k+1}A_{1,n}) = \left(1 - \frac{c}{N}\right)\left\{\pi \Pr(A_{1,n}) - [\pi \Pr(A_{1,n}) - \Pr(A_{1,n+1}A_{1,n})] \cdot \left(1 - \frac{c}{N}\right)^{k-1}\right\} + \pi \frac{c}{N} \Pr(A_{1,n})$$

$$= \pi \Pr(A_{1,n}) - [\pi \Pr(A_{1,n}) - \Pr(A_{1,n+1}A_{1,n})]\left(1 - \frac{c}{N}\right)^{k},$$

as required.

We wish to take the limit of the right side of Eq. 40 as $n \to \infty$ in order to obtain the covariance of the response random variable on any two trials at asymptote. The limits of $\Pr(A_{1,n})$ and $\Pr(A_{1,n+k})$ we know to be equal to π, and from Eq. 35 we have the expression

$$\pi^2 + \pi(1 - \pi)[(1 - c)/N].$$

for the limit of $\Pr(A_{1,n+1}A_{1,n})$. Making the appropriate substitutions in Eq. 40, yields the simple result

$$\lim_{n\to\infty} \operatorname{Cov}(\mathbf{A}_{n+k}\mathbf{A}_n) = \pi^2 - \left[\pi^2 - \pi^2 - \pi(1 - \pi)\frac{(1 - c)}{N}\right]\left(1 - \frac{c}{N}\right)^{k-1} - \pi^2$$

$$= \frac{\pi(1 - \pi)(1 - c)}{N}\left(1 - \frac{c}{N}\right)^{k-1}. \tag{41}$$

Now we are ready to compute $\operatorname{Var}(\bar{\mathbf{A}}_K)$, the variance of A_1-response frequencies in a block of K trials at asymptote, by applying the standard theorem for the variance of a sum of random variables (Feller, 1957):

$$\operatorname{Var}(\bar{\mathbf{A}}_K) = \lim \left[K \operatorname{Var}(\mathbf{A}_n) + 2\sum_{i=1}^{j-1}\sum_{j=2}^{K} \operatorname{Cov}(\mathbf{A}_{n+j}\mathbf{A}_{n+i})\right].$$

Since

$$\lim_{n\to\infty} E(\mathbf{A}_n^2) = \pi \cdot 1 + (1 - \pi) \cdot 0 = \pi,$$

the limiting variance of $\mathbf{A}_n$ is simply

$$\lim_{n\to\infty} \operatorname{Var}(\mathbf{A}_n) = \lim_{n\to\infty} E(\mathbf{A}_n^2) - \lim_{n\to\infty} E(\mathbf{A}_n)^2 = \pi - \pi^2.$$

Substituting this result and that for $\lim \operatorname{Cov}(\mathbf{A}_{n+k}\mathbf{A}_n)$ into the general expression for $\operatorname{Var}(\bar{\mathbf{A}}_K)$, we obtain

$$\operatorname{Var}(\bar{\mathbf{A}}_K) = K\pi(1 - \pi) + 2\sum_{i=1}^{j-1}\sum_{j=2}^{K}\frac{\pi(1 - \pi)(1 - c)}{N}\left(1 - \frac{c}{N}\right)^{j-i-1}$$

$$= K\pi(1 - \pi) + \frac{2\pi(1 - \pi)(1 - c)}{N}\sum_{j=1}^{K}\frac{N}{c}\left[1 - \left(1 - \frac{c}{N}\right)^{j-1}\right]$$

$$= K\pi(1 - \pi) + \frac{2\pi(1 - \pi)(1 - c)}{c}\left\{K - \frac{N}{c}\left[1 - \left(1 - \frac{c}{N}\right)^{K}\right]\right\}. \tag{42}$$

Application of this formula can be conveniently illustrated in terms of the asymptotic data for the 0.8 series. Least-squares determinations of c/N and N from the trigram proportions (using Eq. 34*a-d*) yielded estimates of 0.17 and 1.84, respectively. Inserting these values into Eq. 42, we obtain for a 48-trial block at asymptote $\mathrm{Var}(\bar{\mathbf{A}}_K) = 37.50$; this variance corresponds to a standard deviation of 6.12. The observed standard deviation for the final 48-trial block was 6.94. Thus the theory predicts a variance of the right order of magnitude but, as anticipated, underestimates the observed value.

From the many other statistics that can be derived from the N-element model for two-choice learning data, we take one final example, selected primarily for the purpose of reviewing the technique for deriving sequential statistics. This technique is so generally useful that the major steps should be emphasized: first, expand the desired expression in terms of the conditioning states (as done, for example, in the case of Eq. 30); second, conditionalize responses and reinforcing events on the preceding sequence of events, introducing whatever simplifications are permitted by the boundary conditions of the case under consideration; third, apply the axioms and simplify to obtain the appropriate result. These steps are now followed in deriving an expression of considerable interest in its own right—the probability of an A_1-response following a sequence of exactly ν E_1 reinforcing events:

$$
\begin{aligned}
&\Pr(A_{1,n+\nu} \mid E_{1,n+\nu-1}\cdots E_{1,n}E_{2,n-1}) \\
&\quad= \frac{1}{\pi^{\nu}(1-\pi)} \Pr(A_{1,n+\nu}E_{1,n+\nu-1}\cdots E_{1,n}E_{2,n-1}) \\
&\quad= \frac{1}{\pi^{\nu}(1-\pi)} \sum_{i,j} \Pr(A_{1,n+\nu}C_{i,n+\nu}E_{1,n+\nu-1}\cdots E_{1,n}E_{2,n-1}C_{j,n-1}) \\
&\quad= \frac{1}{\pi^{\nu}(1-\pi)} \sum_{i,j} \Pr(A_{1,n+\nu} \mid C_{i,n+\nu}E_{1,n+\nu-1}\cdots E_{1,n}E_{2,n-1}C_{j,n-1}) \\
&\qquad\qquad \cdot \Pr(C_{i,n+\nu} \mid E_{1,n+\nu-1}\cdots E_{1,n}E_{2,n-1}C_{j,n-1}) \\
&\qquad\qquad \cdot \Pr(E_{1,n+\nu-1}\cdots E_{1,n}E_{2,n-1} \mid C_{j,n-1}) \Pr(C_{j,n-1}) \\
&\quad= \sum_{i,j} \frac{i}{N} \Pr(C_{i,n+\nu} \mid E_{1,n+\nu-1}\cdots E_{1,n}E_{2,n-1}C_{j,n-1}) \Pr(C_{j,n-1}) \\
&\quad= \sum_{j=0}^{N} \left[\left(1 - c\frac{j}{N}\right)\left\{1 - \left(1 - \frac{j}{N}\right)\left(1 - \frac{c}{N}\right)^{\nu}\right\}\right. \\
&\qquad\qquad \left. + c\frac{j}{N}\left\{1 - \left(1 - \frac{j-1}{N}\right)\left(1 - \frac{c}{N}\right)^{\nu}\right\}\right] \Pr(C_{j,n-1}) \\
&\quad= \sum_{j=0}^{N} \left[1 - \left(1 - \frac{j}{N}\right)\left(1 - \frac{c}{N}\right)^{\nu} - c\frac{j}{N}\cdot\frac{1}{N}\left(1 - \frac{c}{N}\right)^{\nu}\right] \Pr(C_{j,n-1}) \\
&\quad= 1 - (1 - p_{n-1})\left(1 - \frac{c}{N}\right)^{\nu} - \frac{c}{N} p_{n-1}\left(1 - \frac{c}{N}\right)^{\nu} \\
&\quad= 1 - \left[1 - \left(1 - \frac{c}{N}\right)p_{n-1}\right]\left(1 - \frac{c}{N}\right)^{\nu}. \qquad (43)
\end{aligned}
$$

The derivation has a formidable appearance, mainly because we have spelled out the steps in more than customary detail, but each step can readily be justified. The first involves simply using the definition of a conditional probability, $\Pr(A \mid B) = \Pr(AB)/\Pr(B)$, together with the fact that in the simple noncontingent case $\Pr(E_{1,n}) = \pi$ and $\Pr(E_{2,n}) = 1 - \pi$ for all n and $\Pr(E_{1,n+\nu-1} \ldots E_{1,n}E_{2,n-1}) = \pi^{\nu}(1 - \pi)$. The second step introduces the conditioning states $C_{i,n+\nu}$ and $C_{j,n-1}$, denoting the states in which i elements are conditioned to A_1 on trial $n + \nu$ and j elements on trial $n - 1$, respectively. Their insertion into the right-hand expression of line 1 is permissible, since the summation of $\Pr(C_i)$ over all values of i is unity and similarly for the summation of $\Pr(C_j)$. The third step is based solely on repeated application of the defining equation for a conditional probability, which permits the expansion

$$\Pr(ABC \ldots J) = \Pr(A \mid BC \ldots J)\Pr(B \mid C \ldots J) \ldots \Pr(J).$$

The fourth step involves assumptions of the model: the conditionalization of $A_{1,n+\nu}$ on the preceding sequence can be reduced to $\Pr(A_{1,n+\nu} \mid C_{i,n+\nu}) = i/N$, since, according to the theory, the preceding history affects response probability on a given trial only insofar as it determines the state of conditioning, that is, the proportion of elements conditioned to the given response. The decomposition of

$$\Pr(E_{1,n+\nu-1} \ldots E_{1,n}E_{2,n-1}C_{j,n-1}) \quad \text{into} \quad \pi^{\nu}(1 - \pi)\Pr(C_{j,n-1})$$

is justified by the special assumptions of the simple noncontingent case. The fifth step involves calculating, for each value of j on trial $n - 1$, the expected proportion of elements conditioned to A_1 on trial $n + \nu$. There are two main branches to the process, starting with state C_j on trial $n - 1$. In one, which by the axioms has probability $1 - c(j/N)$, the state of conditioning is unchanged by the E_2-event on trial $n - 1$; then, applying Eq. 37 with $\pi = 1$ (since from trial n onward we are dealing with a sequence of E_1's) and $\Pr(A_{1,1}) = j/N$, we obtain the expression

$$\{1 - [1 - (j/N)][1 - (c/N)]^{\nu}\}$$

for the expected proportion of elements connected to A_1 on trial $n + \nu$ in this branch. In the other branch, which has probability $c(j/N)$, application of Eq. 37 with $\pi = 1$ and $\Pr(A_{1,1}) = (j - 1)/N$ yields the expression $\{1 - [1 - (j - 1)/N](1 - c/N)^{\nu}\}$ for the expected proportion of elements connected to A_1 on trial $n + \nu$. Carrying out the summation over j and using the by-now familiar property of the model that

$$\sum_{j=0}^{N} \frac{j}{N} \Pr(C_{j,n-1}) = \Pr(A_{1,n-1}) = p_{n-1},$$

we finally arrive at the desired expression for probability of A_1 following exactly ν E_1's.

Application of Eq. 43 can conveniently be illustrated in terms of the 0.8 series. Using the estimate of 0.17 for c/N (obtained previously from the trigram statistics) and taking $p_{n-1} = 0.83$ (the mean proportion of A_1-responses over the last 96 trials of the 0.8 series), we can compute the following values for the conditional response proportions:

ν	0	1	2	3	4
Theoretical	0.689	0.742	0.786	0.822	0.852
Observed	0.695	0.787	0.838	0.859	0.897

It can be seen that the trend of the theoretical values represents quite well the trend of the observed proportions over the last 96 trials. Somewhat surprisingly, the observed proportions run slightly *above* the predicted values. There is no indication here of the "negative recency effect" (decrease in A_1-proportion with increasing length of the E_1-sequence) reported in a number of published two-choice studies (e.g., Jarvik, 1951; Nicks, 1959). It may be significant that no negative recency effect is observed in the 0.8 series, which, it will be recalled, involved well-practiced subjects who had had experience with a wide range of π-values in preceding series. However, the effect *is* observed in the 0.6 series, conducted with subjects new to this type of experiment (cf. Suppes & Atkinson, 1960, pp. 212–213). This differential result appears to support the idea (Estes, 1962) that the negative recency phenomenon is attributable to guessing habits carried over from everyday life to the experimental situation and extinguished during a long training series conducted with noncontingent reinforcement.

We shall conclude our analysis of the N-element pattern model by proving a very general "matching theorem." The substance of this theorem is that, so long as either an E_1 or an E_2 reinforcing event occurs on each trial, the proportion of A_1-responses for any individual subject should tend to match the proportion of E_1-events over a sufficiently long series of trials regardless of the reinforcement schedule.

For purposes of this derivation, we shall identify by a subscript x the probabilities and events associated with the individual x in a population of subjects; thus $p_{x1,n}$ will denote probability of an A_1-response by subject x on trial n, and $\mathbf{E}_{x1,n}$ and $\mathbf{A}_{x1,n}$ will denote random variables which take on the values 1 or 0 according as an E_1-event and an A_1-response do or do not occur in this subject's protocol on trial n. With this notation, the probability of an A_1-response by subject x on trial $n + 1$ can be expressed by the recursion

$$p_{x1,n+1} = p_{x1,n} + \frac{c}{N}(\mathbf{E}_{x1,n} - \mathbf{A}_{x1,n}). \tag{44}$$

The genesis of Eq. 44 should be reasonably obvious if we recall that $p_{x1,n}$ is equal to the proportion of elements currently conditioned to the A_1-response. This proportion can change only if an E_1-event occurs on a trial when a stimulus pattern conditioned to A_2 is sampled, in which case $\mathbf{E}_{x1,n} - \mathbf{A}_{x1,n} = 1 - 0 = 1$, or if an E_2-event occurs on a trial when a pattern conditioned to A_1 is sampled, in which case

$$\mathbf{E}_{x1,n} - \mathbf{A}_{x1,n} = 0 - 1 = -1.$$

In the first case the proportion of patterns conditioned to A_1 increases by $1/N$ if conditioning is effective (which has probability c) and in the second case this proportion decreases by $1/N$ (again with probability c).

Consider now a series of, say, n^* trials: we can convert Eq. 44 into an analogous recursion for response proportions over the series simply by summing both sides over n and dividing by n^*, namely,

$$\frac{1}{n^*}\sum_{n=1}^{n^*} p_{x1,n+1} = \frac{1}{n^*}\sum_{n=1}^{n^*} p_{x1,n} + \frac{1}{n^*}\frac{c}{N}\sum_{n=1}^{n^*}(\mathbf{E}_{x1,n} - \mathbf{A}_{x1,n}).$$

Now we subtract the first sum on the right from both sides of the equation and distribute the second sum on the right to obtain

$$\frac{p_{x1,n+1} - p_{x1,1}}{n^*} = \frac{1}{n^*}\frac{c}{N}\sum_{n=1}^{n^*}\mathbf{E}_{x1,n} - \frac{1}{n^*}\frac{c}{N}\sum_{n=1}^{n^*}\mathbf{A}_{x1,n}.$$

The limit of the left side of this last equation is obviously zero as $n^* \to \infty$; thus taking the limit and rearranging we have[8]

$$\lim_{n^*\to\infty}\frac{1}{n^*}\sum_{n=1}^{n^*}\mathbf{A}_{x1,n} = \lim_{n^*\to\infty}\frac{1}{n^*}\sum_{n=1}^{n^*}\mathbf{E}_{x1,n}. \tag{45}$$

[8] Equation 45 holds only if the two limits exist, which will be the case if the reinforcing event on trial n depends at most on the outcomes of some finite number of preceding trials. When this restriction is not satisfied, a substantially equivalent theorem can be derived simply by dividing both sides of the equation immediately preceding by $\frac{1}{n^*}\sum_{n=1}^{n^*}\mathbf{E}_{x1,n}$ before passing to the limit; that is

$$\frac{p_{x1,n+1} - p_{x1,1}}{\sum_{n=1}^{n^*}\mathbf{E}_{x1,n}} = \frac{c}{N} - \frac{c}{N}\frac{\sum_{n=1}^{n^*}\mathbf{A}_{x1,n}}{\sum_{n=1}^{n^*}\mathbf{E}_{x1,n}}.$$

Except for special cases in which the sum in the denominators converges, the limit of the left-hand side is zero and

$$\lim_{n^*\to\infty}\frac{\sum_{n=1}^{n^*}\mathbf{A}_{x1,n}}{\sum_{n=1}^{n^*}\mathbf{E}_{x1,n}} = 1.$$

To appreciate the strength of this prediction, one should note that it holds for the data of an individual subject starting at any arbitrarily selected point in a learning series, provided only that a sufficiently long block of trials following that point is available for analysis. Further, it holds regardless of the values of the parameters N and c (provided that c is not zero) and regardless of the way in which the schedule of reinforcement may depend on preceding events, the trial number, the subject's behavior, or even events outside the system (e.g., the behavior of another individual in a competitive or cooperative social situation). Examples of empirical applications of this theorem under a variety of reinforcement schedules are to be found in studies reported by Estes (1957a) and Friedman et al. (1960).

2.3 Analysis of a Paired-Comparison Learning Experiment

In order to exhibit a somewhat different interpretation of the axioms of Sec. 2.1, we shall now analyze an experiment involving a paired-comparison procedure. The experimental situation consists of a sequence of discrete trials. There are r objects, denoted A_i ($i = 1$ to r). On each trial two (or more) of these objects are presented to the subject and he is required to choose between them. Once his response has been made the trial terminates with the subject winning or losing a fixed amount of money. The subject's task is to win as frequently as possible. There are many aspects of the situation that can be manipulated by the experimenter; for example, the strategy by which the experimenter makes available certain subsets of objects from which the subjects must choose, the schedule by which the experimenter determines whether the selection of a given object leads to a win or loss, and the amount of money won or lost on each trial.

The particular experiment for which we shall essay a theoretical analysis was reported by Suppes and Atkinson (1960, Chapter 11). The problem for the subjects involved repeated choices from subsets of a set of three objects, which may be denoted A_1, A_2, and A_3. On each trial one of the following subsets of objects was presented: (A_1A_2), (A_1A_3), (A_2A_3), or $(A_1A_2A_3)$. The subject selected one of the objects in the presentation set; then the trial terminated with a win or a loss of a small sum of money. The four presentation sets (A_1A_2), (A_1A_3), (A_2A_3), and $(A_1A_2A_3)$ occurred with equal probabilities over the series of trials. Further, if object A_i was selected on a trial, then with probability λ_i the subject lost and with probability $1 - \lambda_i$ he won the predesignated amount. More complex schedules of reinforcement could be used; of particular interest is a schedule in which the likelihood of a win following the selection of a given object depends on the other available objects in the presentation group. For example, the probability of a win following an A_1 choice could differ, depending on whether the (A_1A_2), (A_1A_3), or $(A_1A_2A_3)$ presentation group occurred. The analysis of these more complex schedules does not introduce new mathematical problems and may be pursued by the same methods we shall use for the simpler case.

Before the axioms of Sec. 2.1 can be applied to the present experiment, we need to provide an interpretation of the stimulus situation confronting the subject from trial to trial. The one we select is somewhat arbitrary and in Sec. 3 alternative interpretations are examined. Of course, discrepancies between predicted and observed quantities will indicate ways in which our particular analysis of the stimulus needs to be modified.

We represent the stimulus display associated with the presentation of the pair of objects (A_iA_j) by a set S_{ij} of stimulus patterns of size N; the triple of objects $(A_1A_2A_3)$ is represented by a set of stimulus patterns S_{123} of size N^*. Thus there are four sets of stimulus patterns, and we assume that the sets are pairwise disjoint (i.e., have no patterns in common). Since, in the model under consideration, the stimulus element sampled on any trial represents the full pattern of stimulation effective on the trial, one might wonder why a given combination of objects, say (A_1A_2), should have more than one element associated with it. It might be remarked in this connection that in introducing a parameter N to represent set size we do not necessarily assume $N > 1$. We simply allow for the possibility that such variations in the situation or different orders of presentation of the same set of objects on different trials might give rise to different stimulus patterns. The assumption that the stimulus patterns associated with a given presentation set are pairwise disjoint does not seem appealing on common-sense grounds; nevertheless, it is of interest to see how far we can go in predicting the data of a paired-comparison learning experiment with the simplified model incorporating this highly restrictive assumption. Even though we cannot attempt to handle the positive and negative transfer effects that must occur between different members of the set of patterns associated with a given combination of objects during learning, we may hope to account for statistics of asymptotic data.

When the pair of objects (A_iA_j) is presented, the subject must select A_i or A_j (i.e., make response A_i or A_j); hence all pattern elements in S_{ij} become conditioned to A_i or A_j. Similarly, all elements in S_{123} become conditioned to A_1, A_2, or A_3. When (A_iA_j) is presented, the subject samples a single pattern from S_{ij} and makes the response to which the pattern is conditioned.

The final step, before applying the axioms of Sec. 2.1, is to provide an interpretation of reinforcing events. Our analysis is as follows: if (A_iA_j) is presented and the A_i-object is selected, then (*a*) the E_i reinforcing event occurs if the A_i-response is followed by a win and (*b*) the E_j-event occurs if the A_i-response is followed by a loss. If $(A_iA_jA_k)$ is presented and the A_i-object is selected, then (*a*) the E_i-event occurs if the A_i-response is followed by a win and (*b*) E_j *or* E_k occurs, the two events having equal probabilities, if the A_i-response is followed by a loss. This collection of rules represents only one way of relating the observable trial outcomes to the hypothetical reinforcing events. For example, when A_i is selected from $(A_iA_jA_k)$ and followed by a loss, rather than having E_j or E_k occur with equal likelihoods one might postulate that they occur with probabilities dependent on the ratio of wins following A_j-responses to wins following A_k-responses over previous trials. Many such variations in the rules of correspondence between trial outcomes and reinforcing events have been explored; these variations become particularly important when the experimenter manipulates the amount of money won or lost, the magnitude of reward in animal studies, and related variables (see Estes, 1960b; Atkinson, 1962; and Suppes & Atkinson, 1960, Chapter 11, for discussions of this point).

In analyzing the model we shall use the following notation:

$A_{i,n}^{(ij)}$ = occurrence of an A_i-response on the *n*th presentation of (A_iA_j) [note that the reference is not to the *n*th trial of the experiment but to the *n*th presentation of (A_iA_j)].

$W_n^{(ij)}$ = a win on the *n*th presentation of (A_iA_j).

$L_n^{(ij)}$ = a loss on the *n*th presentation of (A_iA_j).

We now proceed to derive the probability of an A_i-response on the *n*th presentation of (A_iA_j); namely $\Pr(A_{i,n}^{(ij)})$. First we note that the state of conditioning of a stimulus pattern can change only when it is sampled. Since all of the sets of stimulus patterns are pairwise disjoint, the sequence of trials on which (A_iA_j) is presented forms a learning process that may be studied independently of what happens on other trials (see Axiom C4); that is, the interspersing of other types of trials between the *n*th and $(n + 1)$st presentation of (A_iA_j) has no effect on the conditioning of patterns in set S_{ij}.

We now want to obtain a recursive expression for $\Pr(A_{i,n}^{(ij)})$. This can be done by using the same methods employed in Sec. 2.2. But, to illustrate another approach, we proceed differently in this case.

Let $\Pr(A_{i,n}^{(ij)}) = y_n$ and $\Pr(A_{j,n}^{(ij)}) = 1 - y_n$. The possible changes in y_n are given in Fig. 6. With probability $1 - c$ no change occurs in conditioning, regardless of trial events, hence $y_{n+1} = y_n$; with probability c change can occur. If A_i occurs and is followed by a win, then the sampled element remains conditioned to A_i; however, if a loss occurs, the sampled element (which was conditioned to A_i) becomes conditioned to A_j and thus $y_{n+1} = y_n - 1/N$. If A_j occurs and is followed by a win, then

$y_{n+1} = y_n$; however, if it is followed by a loss, the sampled element (which was conditioned to A_j) becomes conditioned to A_i, hence $y_{n+1} = y_n + 1/N$. Putting these results together, we have

$$y_{n+1} = y_n(1 - c) + y_n[cy_n(1 - \lambda_i)] + \left(y_n - \frac{1}{N}\right)(cy_n\lambda_i) + y_n[c(1 - y_n)(1 - \lambda_j)] + \left(y_n + \frac{1}{N}\right)[c(1 - y_n)\lambda_j],$$

which simplifies to the expression

$$y_{n+1} = y_n\left[1 - \frac{c}{N}(\lambda_i + \lambda_j)\right] + \frac{c}{N}\lambda_j. \tag{46}$$

Solving this difference equation, we obtain

$$\Pr(A_{i,n}^{(ij)}) = \frac{\lambda_j}{\lambda_i + \lambda_j} - \left[\frac{\lambda_j}{\lambda_i + \lambda_j} - \Pr(A_{i,1}^{(ij)})\right]\left[1 - \frac{c}{N}(\lambda_i + \lambda_j)\right]^{n-1}. \tag{47}$$

We now consider $\Pr(A_{i,n}^{(123)})$; for simplicity let $\alpha_n = \Pr(A_{1,n}^{(123)})$, $\beta_n = \Pr(A_{2,n}^{(123)})$, and $1 - \alpha_n - \beta_n = \Pr(A_{3,n}^{(123)})$. The possible changes in α_n are given in Fig. 7. For example, on the bottom branch conditioning is effective and an A_3-response occurs which leads to a loss; hence E_1 or E_2 occur with equal probabilities. But an A_3 followed by E_1 makes

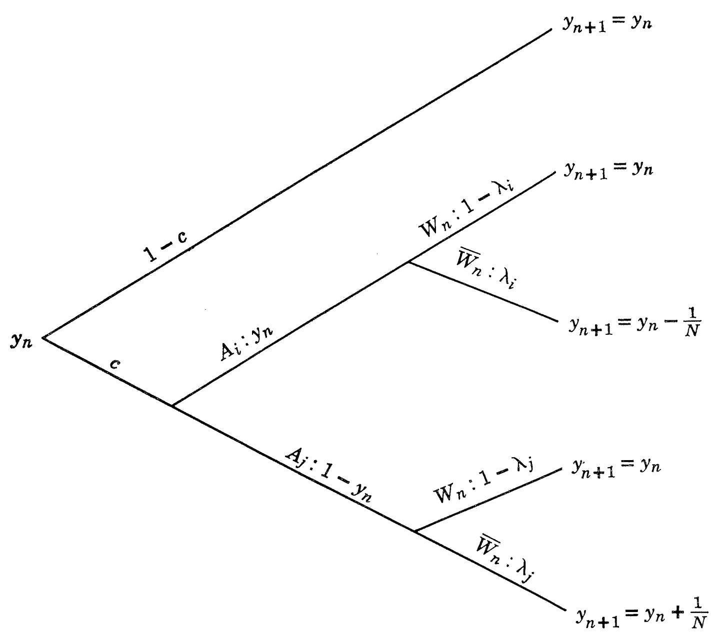

Fig. 6. Branching process for a diad probability on a paired comparison learning trial.

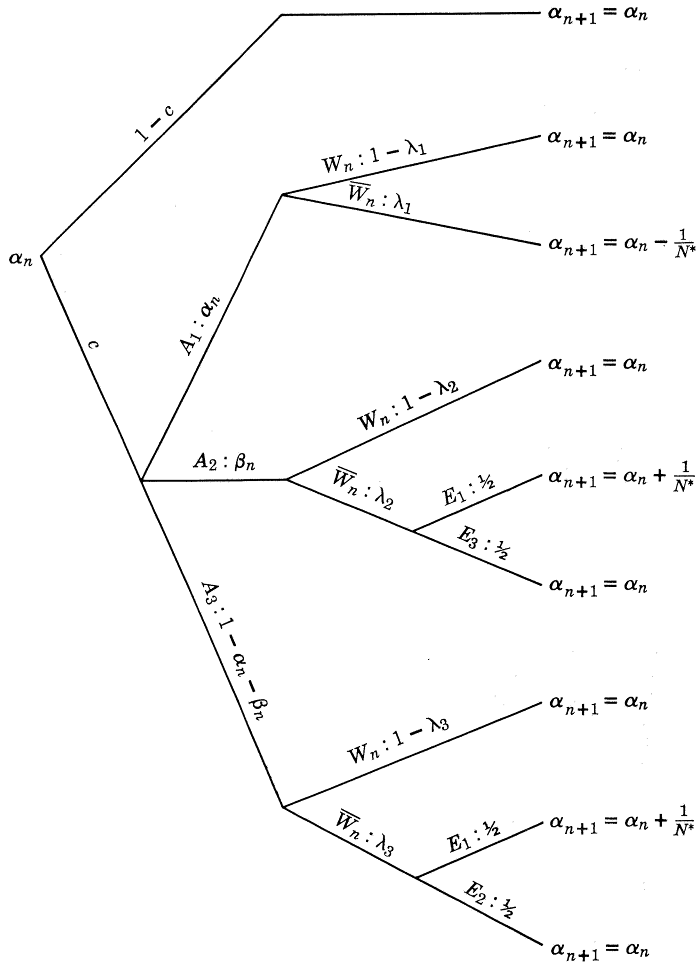

Fig. 7. Branching process for a triad probability on a paired comparison learning trial.

$\alpha_{n+1} = \alpha_n + 1/N$, and A_3 followed by E_2 makes $\alpha_{n+1} = \alpha_n$. Combining the results in this figure yields the following difference equation:

$$\begin{aligned}
\alpha_{n+1} = (1 - c)\alpha_n + \alpha_n[c\alpha_n(1 - \lambda_1)] + \left(\alpha_n - \frac{1}{N^*}\right)(c\alpha_n\lambda_1) \\
+ \alpha_n[c\beta_n(1 - \lambda_2)] + \left(\alpha_n + \frac{1}{N^*}\right)(c\beta_n\lambda_2 \tfrac{1}{2}) + \alpha_n(c\beta_n\lambda_2 \tfrac{1}{2}) \\
+ \alpha_n[c(1 - \alpha_n - \beta_n)(1 - \lambda_3)] + \left(\alpha_n + \frac{1}{N^*}\right)[c(1 - \alpha_n - \beta_n)\lambda_3 \tfrac{1}{2}] \\
+ \alpha_n[c(1 - \alpha_n - \beta_n)\lambda_3 \tfrac{1}{2}].
\end{aligned}$$

Simplifying this result, we obtain

$$\alpha_{n+1} = \alpha_n\left[1 - \frac{c}{2N^*}(2\lambda_1 + \lambda_3)\right] + \beta_n \frac{c}{2N^*}(\lambda_2 - \lambda_3) + \frac{c}{2N^*}\lambda_3. \quad (48a)$$

By a similar argument we obtain

$$\beta_{n+1} = \beta_n\left[1 - \frac{c}{2N^*}(2\lambda_2 + \lambda_3)\right] + \alpha_n \frac{c}{2N^*}(\lambda_1 - \lambda_3) + \frac{c}{2N^*}\lambda_3. \quad (48b)$$

Solutions for the pair of difference equations given by Eqs. 48*a* and 48*b* are well known and can be obtained by a number of different techniques (see Goldberg, 1958, pp. 130–133, or Jordan, 1950). Any solution presented can be verified by substituting into the appropriate difference equations. However for now we shall limit consideration to asymptotic results. In terms of the Markov chain property of our process it can be shown that the limits $\alpha = \lim_{n\to\infty} \alpha_n$ and $\beta = \lim_{n\to\infty} \beta_n$ exist. Letting $\alpha_{n+1} = \alpha_n = \alpha$ and $\beta_{n+1} = \beta_n = \beta$ in Eqs. 48*a* and 48*b*, we obtain

$$\alpha(2\lambda_1 + \lambda_3) = \beta(\lambda_2 - \lambda_3) + \lambda_3$$
$$\beta(2\lambda_2 + \lambda_3) = \alpha(\lambda_1 - \lambda_3) + \lambda_3.$$

Solving for α and β and rewriting, we have

$$\lim_{n\to\infty} \Pr(A_{1,n}^{(123)}) = \frac{\lambda_2\lambda_3}{\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3}, \quad (49a)$$

$$\lim_{n\to\infty} \Pr(A_{2,n}^{(123)}) = \frac{\lambda_1\lambda_3}{\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3}, \quad (49b)$$

and

$$\lim_{n\to\infty} \Pr(A_{3,n}^{(123)}) = \frac{\lambda_1\lambda_2}{\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3}. \quad (49c)$$

The other moments of the distribution of response probabilities can be obtained by following the methods employed in Sec. 2.1; and at asymptote we can generate the entire distribution. In particular, for set S_{ij} the asymptotic probability that k patterns are conditioned to A_i and $N - k$ to A_j is simply

$$\binom{N}{k}\left(\frac{\lambda_j}{\lambda_i + \lambda_j}\right)^k\left(\frac{\lambda_i}{\lambda_i + \lambda_j}\right)^{N-k}.$$

For the set S_{123} the asymptotic probability of k_1 patterns conditioned to A_1, k_2 to A_2, and k_3 to A_3 (where $k_1 + k_2 + k_3 = N^*$) is

$$\frac{N^*!}{k_1!k_2!k_3!}\left(\frac{1}{\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3}\right)^{N^*}(\lambda_2\lambda_3)^{k_1}(\lambda_1\lambda_3)^{k_2}(\lambda_1\lambda_2)^{k_3}.$$

In analyzing data it is helpful also to examine the marginal limiting probability of an A_i-response, $\Pr(A_i)$, in addition to the other quantities already mentioned. We define $\Pr(A_i)$ as the probability of an A_i-response on any trial (regardless of the stimulus display) once the process has reached asymptote. Theoretically

$$\Pr(A_1) = \Pr(A_{1,\infty}^{(12)})\Pr(D^{(12)}) + \Pr(A_{1,\infty}^{(13)})\Pr(D^{(13)}) + \Pr(A_{1,\infty}^{(123)})\Pr(D^{(123)}),$$

$$\Pr(A_2) = \Pr(A_{2,\infty}^{(12)})\Pr(D^{(12)}) + \Pr(A_{2,\infty}^{(23)})\Pr(D^{(23)}) + \Pr(A_{2,\infty}^{(123)})\Pr(D^{(123)}),$$

and

$$\Pr(A_3) = 1 - \Pr(A_1) - \Pr(A_2),$$

where $\Pr(D^{(ij)})$ is the probability of presenting the pair of objects (A_iA_j).

The experimental results we consider were reported in preliminary form in Suppes & Atkinson (1960). Two groups, each involving 48 subjects, were run; subjects in one group won or lost one cent on each trial, and those in the other group won or lost five cents on each trial. We shall consider only the one-cent group, for an analysis of the differential effects of the two reward values requires a more elaborate interpretation of reinforcing events. Subjects were run for 400 trials with the following reinforcement schedule:

$$\lambda_1 = \tfrac{1}{3}, \qquad \lambda_2 = \tfrac{6}{10}, \qquad \lambda_3 = \tfrac{8}{10}.$$

Figure 8 presents the observed proportions of A_1-, A_2-, and A_3-responses in successive 20-trial blocks. The three curves appear to be stable over the last 10 or so blocks; consequently we treat the data over trials 301 to 400 as asymptotic.

By Eq. 47 and Eq. 49*a-c* we may generate predictions for $\Pr(A_{i,\infty}^{(ij)})$ and $\Pr(A_{i,\infty}^{(123)})$. Given these values and the fact that the four presentation sets occur with equal probabilities, we may, as previously shown, generate

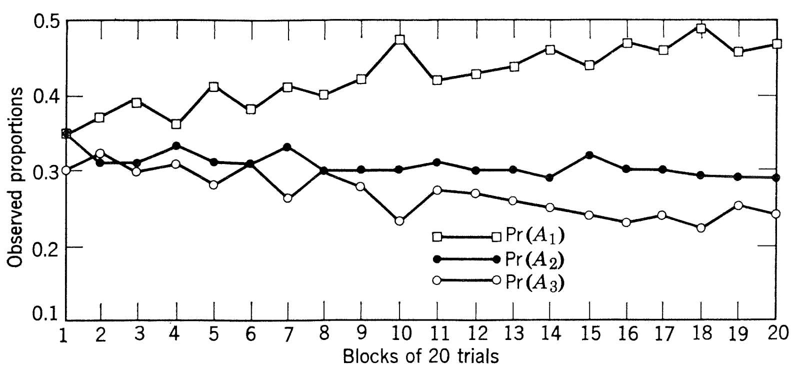

Fig. 8. Observed proportion of A_i-responses in successive 20-trial blocks for paired comparison experiment.

predictions for $\Pr(A_{i,\infty})$. The predicted values for these quantities and the observed proportions over the last 100 trials are presented in Table 4. The correspondence between predicted and observed values is very good, particularly for $\Pr(A_{i,\infty})$ and $\Pr(A_{i,\infty}^{(ij)})$. The largest discrepancy is for the triple presentation set, in which we note that the observed value of $\Pr(A_{1,\infty}^{(123)})$ is 0.041 above the predicted value of 0.507. The statistical problem of determining whether this particular difference is significant is a complex matter and we shall not undertake it here. However, it

Table 4 Theoretical and Observed Asymptotic Choice Proportions for Paired-Comparison Learning Experiment

	Predicted	Observed
$\Pr(A_1)$	0.464	0.473
$\Pr(A_2)$	0.302	0.294
$\Pr(A_3)$	0.234	0.233
$\Pr(A_1^{(12)})$	0.643	0.651
$\Pr(A_1^{(13)})$	0.706	0.700
$\Pr(A_2^{(23)})$	0.571	0.561
$\Pr(A_1^{(123)})$	0.507	0.548
$\Pr(A_2^{(123)})$	0.282	0.258
$\Pr(A_3^{(123)})$	0.211	0.194

should be noted that similar discrepancies have been found in other studies dealing with three or more responses (see Gardner, 1957; Detambel, 1955), and it may be necessary, in subsequent developments of the theory, to consider some reinterpretation of reinforcing events in the multiple-response case.

In order to make predictions for more complex aspects of the data, it is necessary to obtain estimates of c, N, and N^*. Estimation procedures of the sort referred to in Sec. 2.2 are applicable, but the analysis becomes tedious and such details are not appropriate here. However, some comparisons can be made between sequential statistics that do not depend on parameter values. For example, certain nonparametric comparisons can be made between statistics where each depends on c and N but where the difference is independent of these parameters. Such comparisons are particularly helpful when they permit us to discriminate among different models without introducing the complicating factor of having to estimate parameters.

To indicate the types of comparisons that are possible, we may consider the subsequence of trials on which (A_1A_2) is presented and, in particular, the expression

$$\Pr(A_{1,n+1}^{(12)} \mid W_n^{(12)} A_{1,n}^{(12)} W_{n-1}^{(12)} A_{2,n-1}^{(12)});$$

that is, the probability of an A_1-response on the $(n + 1)$st presentation of (A_1A_2), given that on the nth presentation of (A_1A_2) an A_1 occurred and was followed by a win and that on the $(n - 1)$st presentation of (A_1A_2) an A_2 occurred, followed by a win. To compute this probability, we note that

$$\Pr(A_{1,n+1}^{(12)} \mid W_n^{(12)}A_{1,n}^{(12)}W_{n-1}^{(12)}A_{2,n-1}^{(12)}) = \frac{\Pr(A_{1,n+1}^{(12)}W_n^{(12)}A_{1,n}^{(12)}W_{n-1}^{(12)}A_{2,n-1}^{(12)})}{\Pr(W_n^{(12)}A_{1,n}^{(12)}W_{n-1}^{(12)}A_{2,n-1}^{(12)})}.$$

Now our problem is to compute the two quantities on the right-hand side of this equation. We first observe that

$$\Pr(A_{1,n+1}^{(12)}W_n^{(12)}A_{1,n}^{(12)}W_{n-1}^{(12)}A_{2,n-1}^{(12)}) = \sum_{i,j} \Pr(A_{1,n+1}^{(12)}C_{j,n+1}^{(12)}W_n^{(12)}A_{1,n}^{(12)}W_{n-1}^{(12)}A_{2,n-1}^{(12)}C_{i,n-1}^{(12)}),$$

where $C_{i,n}^{(12)}$ denotes the conditioning state for set S_{12} in which i elements are conditioned to A_1 and $N - i$ to A_2 on the nth presentation of (A_1A_2). Conditionalizing and applying the axioms, we may expand the last expression into

$$\sum_{i,j} \Pr(A_{1,n+1}^{(12)} \mid C_{j,n+1}^{(12)}) \Pr(C_{j,n+1}^{(12)} \mid W_n^{(12)}A_{1,n}^{(12)}W_{n-1}^{(12)}A_{2,n-1}^{(12)}C_{i,n-1}^{(12)}) \cdot (1 - \lambda_1) \Pr(A_{1,n}^{(12)} \mid W_{n-1}^{(12)}A_{2,n-1}^{(12)}C_{i,n-1}^{(12)})(1 - \lambda_2) \cdot \Pr(A_{2,n-1}^{(12)} \mid C_{i,n-1}^{(12)}) \Pr(C_{i,n-1}^{(12)}).$$

Further, the sampling and response axioms permit the simplifications

$$\Pr(A_{1,n+1}^{(12)} \mid C_{j,n+1}^{(12)}) = \frac{j}{N},$$

$$\Pr(A_{1,n}^{(12)} \mid W_{n-1}^{(12)}A_{2,n-1}^{(12)}C_{i,n-1}^{(12)}) = \frac{i}{N},$$

and

$$\Pr(A_{2,n-1}^{(12)} \mid C_{i,n-1}^{(12)}) = \frac{N - i}{N}.$$

Finally, in order to carry out the summation, we make use of the relation

$$\Pr(C_{j,n+1}^{(12)} \mid W_n^{(12)}A_{1,n}^{(12)}W_{n-1}^{(12)}A_{2,n-1}^{(12)}C_{i,n-1}^{(12)}) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j, \end{cases}$$

which expresses the fact that no change in the conditioning state can occur if the pattern sampled leads to a win (see Axiom C2). Combining these results and simplifying, we have

$$\Pr(A_{1,n+1}^{(12)}W_n^{(12)}A_{1,n}^{(12)}W_{n-1}^{(12)}A_{2,n-1}^{(12)}) = (1 - \lambda_1)(1 - \lambda_2) \sum_i \left(\frac{i}{N}\right)^2 \left(\frac{N - i}{N}\right) \Pr(C_{i,n-1}^{(12)}). \quad (50a)$$

Similarly, we obtain

$$\Pr(W_n^{(12)}A_{1,n}^{(12)}W_{n-1}^{(12)}A_{2,n-1}^{(12)}) = (1-\lambda_1)(1-\lambda_2)\sum_i \frac{i}{N}\left(\frac{N-i}{N}\right)\Pr(C_{i,n-1}^{(12)}), \quad (50b)$$

and, finally, taking the quotient of the last two expressions,

$$\Pr(A_{1,n+1}^{(12)} \mid W_n^{(12)}A_{1,n}^{(12)}W_{n-1}^{(12)}A_{2,n-1}^{(12)}) = \frac{\sum_i \left(\frac{i}{N}\right)^2\left(\frac{N-i}{N}\right)\Pr(C_{i,n-1}^{(12)})}{\sum_i \frac{i}{N}\left(\frac{N-i}{N}\right)\Pr(C_{i,n-1}^{(12)})}. \quad (50c)$$

We next consider the same sequential statistic but with the responses reversed on trials n and $n-1$; namely,

$$\Pr(A_{1,n+1}^{(12)} \mid W_n^{(12)}A_{2,n}^{(12)}W_{n-1}^{(12)}A_{1,n-1}^{(12)})$$

Interestingly enough, if we compute

$$\Pr(A_{1,n+1}^{(12)}W_n^{(12)}A_{2,n}^{(12)}W_{n-1}^{(12)}A_{1,n-1}^{(12)})$$

and

$$\Pr(W_n^{(12)}A_{2,n}^{(12)}W_{n-1}^{(12)}A_{1,n-1}^{(12)}),$$

they turn out to be expressed by the right sides of Eq. 50*a* and 50*b*, respectively. Hence, for all n,

$$\Pr(A_{1,n+1}^{(12)} \mid W_n^{(12)}A_{1,n}^{(12)}W_{n-1}^{(12)}A_{2,n-1}^{(12)}) = \Pr(A_{1,n+1}^{(12)} \mid W_n^{(12)}A_{2,n}^{(12)}W_{n-1}^{(12)}A_{1,n-1}^{(12)}). \quad (51)$$

Comparable predictions, of course, hold for the subsequences of trials on which (A_1A_3) or (A_2A_3) are presented.

Equation 51 provides a test of the theory which does not depend on parameter estimates. Further, it is a prediction that differentiates between this model and many other models. For example, in the next section we consider a certain class of linear models, and it can be shown that they generate the same predictions for the quantities in Table 4 as the pattern model. However, the sequential equality displayed in Eq. 51 does not hold for the linear model.

To check these predictions, we shall utilize the data over all trials of the (A_1A_2) subsequence and not restrict the analysis to asymptotic performance. Specifically, we define

$$\zeta_{112} = \sum_n \Pr(A_{1,n+1}^{(12)}W_n^{(12)}A_{1,n}^{(12)}W_{n-1}^{(12)}A_{2,n-1}^{(12)})$$

$$\zeta_{121} = \sum_n \Pr(A_{1,n+1}^{(12)}W_n^{(12)}A_{2,n}^{(12)}W_{n-1}^{(12)}A_{1,n-1}^{(12)})$$

$$\zeta_{12} = \sum_n \Pr(W_n^{(12)}A_{1,n}^{(12)}W_{n-1}^{(12)}A_{2,n-1}^{(12)})$$

$$\zeta_{21} = \sum_n \Pr(W_n^{(12)}A_{2,n}^{(12)}W_{n-1}^{(12)}A_{1,n-1}^{(12)}).$$

But by the results just obtained we have $\zeta_{121} = \zeta_{112}$ and $\zeta_{21} = \zeta_{12}$ for any given subject. Further, if we define $\dot{\zeta}_{ijk}$ as the sum of the ζ_{ijk}'s over all subjects, then it follows that $\dot{\zeta}_{121} = \dot{\zeta}_{112}$, independent of intersubject differences in c and N. Similarly, $\dot{\zeta}_{12} = \dot{\zeta}_{21}$. Thus we have a set of predictions that are not only nonparametric but that require no restrictive assumptions on variability between subjects. Observed frequencies corresponding to these theoretical quantities are as follows:

$$\dot{\zeta}_{121} = 140 \qquad \dot{\zeta}_{112} = 138$$

$$\dot{\zeta}_{21} = 243 \qquad \dot{\zeta}_{12} = 244$$

$$\frac{\dot{\zeta}_{121}}{\dot{\zeta}_{21}} = 0.576 \qquad \frac{\dot{\zeta}_{112}}{\dot{\zeta}_{12}} = 0.566.$$

Similarly, for the (A_1A_3) subsequence,

$$\dot{\zeta}_{131} = 67 \qquad \dot{\zeta}_{113} = 64$$

$$\dot{\zeta}_{31} = 120 \qquad \dot{\zeta}_{13} = 122$$

$$\frac{\dot{\zeta}_{131}}{\dot{\zeta}_{31}} = 0.558 \qquad \frac{\dot{\zeta}_{113}}{\dot{\zeta}_{13}} = 0.525.$$

Finally, for the (A_2A_3) subsequence,

$$\dot{\zeta}_{232} = 45 \qquad \dot{\zeta}_{223} = 49$$

$$\dot{\zeta}_{32} = 82 \qquad \dot{\zeta}_{23} = 87$$

$$\frac{\dot{\zeta}_{232}}{\dot{\zeta}_{32}} = 0.549 \qquad \frac{\dot{\zeta}_{223}}{\dot{\zeta}_{23}} = 0.563.$$

Further analyses will be required to determine whether the pattern model gives an entirely satisfactory interpretation of paired-comparison learning. It is already apparent, however, that it may be difficult indeed to find another theory with equally simple machinery that will take us further in this direction than the pattern model.

3. A COMPONENT MODEL FOR STIMULUS COMPOUNDING AND GENERALIZATION

3.1 Basic Concepts; Conditioning and Response Axioms

In the preceding section we simplified our analysis of learning in terms of the N-element pattern model by assuming that all of the patterns involved in a given experiment are disjoint or, at any rate, that generalization effects from one stimulus pattern to another are negligible. Now we shall go to the other extreme and treat problems of simple transfer of training between different stimulus situations that have elements in common, and make no reference to a learning process occurring over trials. Again the basic mathematical apparatus is that of sets and elements but with a reinterpretation that needs to be clearly distinguished from that of the pattern model. In Secs. 1 and 2 we regarded the pattern of stimulation effective on any trial as a single element sampled from a larger set of such patterns; now we shall consider the trial pattern as itself constituting a set of elements, the elements representing the various components or aspects of the stimulus situation that may be sampled by the subject in differing combinations on different trials. We proceed first to give the two basic axioms that establish the dependence of response probability on the conditioning state of the stimulus sample. Then some theorems that specify relationships between response probabilities in overlapping stimulus samples are derived and are illustrated in terms of applications to experiments on simple stimulus compounding. Consideration of the process by which trial samples are drawn from a larger stimulus population is deferred to Sec. 3.3.

The basic axioms of the component model are as follows:

Basic Axioms

C1. *The sample s of stimulation effective on any trial is partitioned into subsets s_i ($i = 1, 2, \ldots r$, where r is the number of response alternatives), the ith subset containing the elements conditioned to (or "connected to") response A_i.*

C2. *The probability of response A_i in the presence of the stimulus sample s is given by*

$$\Pr(A_i \mid s) = \frac{N(s_i)}{N(s)},$$

where $N(x)$ denotes the number of elements in the set x.

In Axiom C1 we modify the usual definition of a partition to the extent of permitting some of the subsets to be empty; that is, there may be some response alternatives that are conditioned to none of the elements of s. We do mean to assume, however, that each element of s is conditioned to exactly one response. The substance of Axiom C2 is, then, to make the probability that a given response will be evoked by s equal to the proportion of elements of s that are conditioned to that response.

3.2 Stimulus Compounding

An elementary transfer situation arises if two responses are reinforced, each in the presence of a different stimulus sample, and all or part of one sample is combined with all or part of the other to form a new test situation. To begin with a special case, let us consider an experiment conducted in the laboratory of one of the writers (W.K.E.).[9] In one stage of the experiment a number of disjoint samples of three distinct cues drawn from a large population were used as the stimulus members of paired-associate items, and by the usual method of paired presentation one response was reinforced in the presence of some of these samples and a different response in the presence of others. The constituent cues, intended to serve as the empirical counterparts of stimulus elements, were various typewriter symbols, which for present purposes we designate by small letters a, b, c, etc.; the responses were the numbers "one" and "two," spoken aloud. Instructions to the subjects indicated that the cues represented symptoms and the numbers diseases with which the symptoms were associated. Following the training trials, new combinations of "symptoms" were formed, and the subjects were instructed to make their best guesses at the correct diagnoses.

Suppose now that response A_1 had been reinforced in the presence of the sample (abc) and response A_2 in the presence of the sample (def). If a test trial were given subsequently with the sample (abd), direct application of Axiom C2 yields the prediction that response A_1 should occur with probability $\frac{2}{3}$. Similarly, if a test were given with the sample (ade), response A_1 would be predicted to occur with probability $\frac{1}{3}$. Results obtained with 40 subjects, each given 24 tests of each type, were as follows:

percentage overlap of training and test sets	0.667	0.333
percentage response 1 to test set	0.669	0.332

Success in bringing off a priori predictions of this sort depends not only on the basic soundness of the theory but also on one's success in realizing various simplifying assumptions in the experimental situation. As we have mentioned, it was our intention in designing this experiment to choose cues, a, b, c, etc., which would take on the role of stimulus elements. Actually, in order to justify our theoretical predictions, it was necessary only that the cues behave as equal-sized sets of elements. To bring out the

[9] This experiment was conducted at Indiana University with the assistance of Miss Joan SeBreny.

importance of the equal N assumption, let us suppose that the individual cues actually correspond to sets s_a, s_b, etc., of elements. Then, given the same training (response A_1 reinforced to the combination *abc* and response A_2 to *def*) and assuming the training effective in conditioning all elements of each subset to the reinforced response, application of Axiom C2 yields for the probability of response A_1 to *abd*

$$\Pr(A_1 \mid s_a s_b s_d) = \frac{N_a + N_b}{N_a + N_b + N_d},$$

where we have used the obvious abbreviation $N(s_i) = N_i$. This equation reduces to $\Pr(A_1 \mid s_a s_b s_d) = \frac{2}{3}$ if $N_a = N_b = N_d$.

In this experiment we depended on common-sense considerations to choose cues that could be expected to satisfy the equal-N requirement and also counterbalanced the design of the experiment so that minor deviations might be expected to average out. Sometimes it may not be possible to depend on common-sense considerations. In that case a preliminary experiment can be utilized to check on the simplifying assumptions. Suppose, for example, we had been in doubt as to whether cues *a* and *b* would behave as equal-sized sets. To check on them, we could have run a preliminary experiment in which we reinforced, say, response A_1 to *a* and response A_2 to *b*, then tested with the compound *ab*. Probability of response A_1 to *ab* is, according to the model, given by

$$\Pr(A_1 \mid s_a s_b) = \frac{N_a}{N_a + N_b},$$

which should deviate in the appropriate direction from $\frac{1}{2}$ if N_a and N_b are not equal. By means of calibration experiments of this sort sets of cues satisfying the equal-N assumption can be assembled for use in further research involving applications of the model.

The expressions we have obtained for probabilities of response to stimulus compounds can readily be generalized with respect both to set sizes and to level of training. Suppose that a collection of cues *a*, *b*, *c*, . . . corresponds to a collection of stimulus sets s_a, s_b, s_c, . . . of sizes N_a, N_b, N_c, . . . and that some response A_j is conditioned to a proportion p_{aj} of the elements in s_a, a proportion p_{bj} of the elements in s_b, and so on. Then probability of response A_j to a compound of these cues is, by Axiom C2, expressed by the relation

$$\Pr(A_j \mid s_a, s_b, s_c, \ldots) = \frac{N_a p_{aj} + N_b p_{bj} + N_c p_{cj} + \cdots}{N_a + N_b + N_c + \cdots}. \tag{52}$$

Application of Eq. 52 can be illustrated in terms of a study of probabilistic discrimination learning reported in Estes, Burke, Atkinson, & Frankmann (1957). In this study the individual cues were lights that differed from each other only in their positions on a panel. The first stage of the experiment consisted in discrimination training according to a routine that we shall not describe here except to say that on theoretical grounds it was predicted that at the end of training the proportion of elements in a sample associated with the ith light conditioned to the first of two alternative responses would be given by $p_{i1} = i/13$. Following this training, the subjects were given compounding tests with various triads of lights. Considering, say, the triad of lights 1, 2, and 3, the values of p_{i1} should be $p_{11} = \frac{1}{13}$, $p_{21} = \frac{2}{13}$, and $p_{31} = \frac{3}{13}$, assuming $N_1 = N_2 = N_3 = N$, and substituting these values into Eq. 52, we obtain

$$\Pr(A_1 \mid 1, 2, 3) = \frac{N/13 + 2N/13 + 3N/13}{3N} = \frac{2}{13} = 0.15$$

as the predicted probability of response 1 to the compound 1, 2, 3. Theoretical values similarly computed for a number of triads are compared with the empirical test proportions reported by Estes et al. in Table 5.

Table 5 Theoretical and Observed Proportions of Response A_1 to Triads of Lights in Stimulus Compounding Test

Triad	Theoretical	Observed
1, 2, 3	0.15	0.22
4, 5, 6	0.38	0.31
1, 3, 11	0.38	0.41
7, 8, 9	0.62	0.59
2, 10, 12	0.62	0.58
10, 11, 12	0.85	0.77

An important consideration in applications of models for stimulus compounding is the question whether the experimental situation contains an appreciable amount of background stimulation in addition to the controlled stimuli manipulated by the experimenter. Suppose, for example, we are interested in the problem that a compound of two conditioned stimuli, say a light and a tone, each of which has been paired with the same unconditioned stimulus, may have a higher probability of evoking a conditioned response (CR) than either of the stimuli presented separately. To analyze this problem in terms of the present model, we may represent the light and the tone by stimulus sets s_L and s_T. Assuming that as a result of the previous reinforcement the proportions of conditioned elements in s_L and s_T (and therefore the probabilities of CR's to the stimuli taken separately) are p_L and p_T, respectively, application of Axiom C2 yields for the probability of a CR to the compound of light and tone presented together, neglecting any possible background stimulation,

$$\Pr(CR \mid L, T) = \frac{N_L p_L + N_T p_T}{N_L + N_T}.$$

Clearly, the probability of a *CR* to the compound is simply a weighted mean of p_L and p_T, and therefore its value must fall between the probabilities of a *CR* to the two conditioned stimuli taken separately. No "summation" effect is predicted.

Often, however, it may be unrealistic to assume that background stimulation from the apparatus and surroundings is negligible. In fact, the experimenter may have to count on an appreciable amount of background stimulation, predominantly conditioned to behaviors incompatible with the *CR*, to prevent "spontaneous" occurrences of the to-be-conditioned response during intervals between presentations of the experimentally controlled stimuli. Let us now expand our representation of the conditioning situation by defining a set s_b of background elements, a proportion p_b of which are conditioned to the *CR*. For simplicity, we shall consider only the special case of $p_b = 0$. Then the theoretical probabilities of evocation of the *CR* by the light, the tone, and the compound of light and sound (together with background stimulation in each case) are given by

$$\Pr(CR \mid L) = \frac{N_L p_L}{N_L + N_b},$$

$$\Pr(CR \mid T) = \frac{N_T p_T}{N_T + N_b},$$

and

$$\Pr(CR \mid L, T) = \frac{N_T p_T + N_L p_L}{N_T + N_L + N_b},$$

respectively. Under these conditions it is possible to obtain a summation effect. Assume, for example, that $N_T = N_L = N_b$ and $p_T > p_L$, so $\Pr(CR \mid T) > \Pr(CR \mid L)$. Taking the difference between the probability of a *CR* to the compound and probability of a *CR* to the tone alone, we have

$$\begin{aligned}\Pr(CR \mid L, T) - \Pr(CR \mid T) &= \frac{p_T + p_L}{3} - \frac{p_T}{2} \\ &= \frac{2p_T + 2p_L - 3p_T}{6} \\ &= \frac{2p_L - p_T}{6},\end{aligned}$$

which is positive if the inequality $2p_L > p_T$ holds. Thus, in this case, probability of a *CR* to the compound will exceed probability of a *CR* to either conditioned stimulus alone, provided that p_T is not more than twice p_L.

The role of background stimuli has been particularly important in the interpretation of drive stimuli. It has been assumed (Estes, 1958, 1961a) that in simple animal learning experiments (e.g., those involving the learning of running or bar-pressing responses with food or water reward) the stimulus sample to which the animal responds at any time is compounded from several sources: the experimentally controlled conditioned stimulus (*CS*) or equivalent; stimuli, perhaps largely intraorganismic in origin, controlled by the level of food or water deprivation; and extraneous stimuli that are not systematically correlated with reward of the response undergoing training and therefore remain for the most part connected to competing responses. It is assumed further that the sizes of samples of elements associated with the *CS* and with extraneous sources s_C and s_E are independent of drive but that the size of the sample of drive-stimulus elements, s_D, increases as a function of deprivation. In most simple reward-learning experiments conditioning of the *CS* and drive cues would proceed concurrently, and it might be expected that at a given stage of learning the proportions of elements in samples from these sources conditioned to the rewarded response *R* would be equal, that is, $p_C = p_D$. If this were the case, then probability of the rewarded response would be independent of deprivation; for, letting D and D' correspond to levels of deprivation such that $N_D < N_{D'}$, we have as the theoretical probabilities of response *R* at the two deprivations,

$$\Pr(R \mid CS, D) = \frac{N_C p_C + N_D p_D}{N_C + N_D}$$

and

$$\Pr(R \mid CS, D') = \frac{N_C p_C + N_{D'} p_{D'}}{N_C + N_{D'}}.$$

If the same training were given at the two drive levels, then we would have $p_D = p_{D'}$ as well as $p_C = p_D$; in this case the difference between the two expressions is zero. Considering the same assumptions, but with extraneous cues taken explicitly into account, we arrive at a quite different picture. In this case the two expressions for response probability are

$$\Pr(R \mid CS, D, E) = \frac{N_C p_C + N_D p_D + N_E p_E}{N_C + N_D + N_E}$$

and

$$\Pr(R \mid CS, D', E) = \frac{N_C p_C + N_{D'} p_{D'} + N_E p_E}{N_C + N_{D'} + N_E}.$$

Now, letting $p_C = p_D = p_{D'} = p$ and, for simplicity, taking $p_E = 0$, we obtain for the difference

$$\Pr(R \mid CS, D', E) - \Pr(R \mid CS, D, E)$$
$$= p\left[\frac{N_C + N_{D'}}{N_C + N_{D'} + N_E} - \frac{N_C + N_D}{N_C + N_D + N_E}\right]$$
$$= p\frac{N_E(N_{D'} - N_D)}{(N_C + N_{D'} + N_E)(N_C + N_D + N_E)},$$

which is obviously greater than zero, given the assumption $N_{D'} > N_D$. Thus, in this theory, the principal reason why probability of the rewarded response tends, other things being equal, to be higher at higher deprivations is that the larger the sample of drive stimuli, the more effective it is in outweighing the effects of extraneous stimuli.

3.3 Sampling Axioms and Major Response Theorem of Fixed Sample Size Model

In Sec. 3.2 we considered some transfer effects which can be derived within a component model by considering only relationships among stimulus samples that have had different reinforcement histories. Generally, however, it is desirable to take account of the fact that there may not always be a one-to-one correspondence between the experimental stimulus display and the stimulation actually influencing the subject's behavior. Because of a number of factors, for example, variations in receptor-orienting responses, fluctuations in the environmental situation, or variations in excitatory states or thresholds of receptors, the subject often may sample only a portion of the stimulation made available by the experimenter. One of the chief problems of statistical learning theories has been to formulate conceptual representations of the stimulus sampling process and to develop their implications for learning phenomena. With respect to specific mathematical properties of the sampling process, component models that have appeared in the literature may be classified into two main types: (1) models assuming fixed sampling probabilities for the individual elements of a stimulus population, in which case sample size varies randomly from trial to trial; and (2) models assuming a fixed ratio between sample size and population size. The first type was first discussed by Estes and Burke (1953), the second by Estes (1950), and some detailed comparisons of the two types have been presented by Estes (1959b). In this section we shall limit consideration to models of the second type, since these are in most respects easier to work with.

In the remainder of this section we shall distinguish stimulus populations and samples by using S, with subscripts as needed, for a population and s for a sample. The sampling axioms to be utilized are as follows:

Sampling Axioms

S1. *For any fixed, experimenter-defined stimulating situation, sample size and population size are constant over trials.*
S2. *All samples of the same size have equal probabilities.*

A prerequisite to nearly all applications of the model is a theorem relating response probability to the state of conditioning of a stimulus population. We derive this theorem in terms of a stimulus situation S containing N elements from which a sample of size $N(s) = \sigma$ is drawn on each trial. Assuming that some number N_i of the elements of S is conditioned to response A_i, we wish to obtain an expression for the expected proportion of elements conditioned to A_i in samples drawn from S, since this proportion will, by Axiom C2, be equal to the probability of evocation of response A_i by samples from S. We begin, as usual, with the probability in which we are interested; then, using the axioms of the model as appropriate, we proceed to expand in terms of the state of conditioning and possible stimulus samples:

$$\Pr(A_i \mid S) = \sum_s \Pr(A_i \mid s) \Pr(s \mid S),$$

the summation being over all samples of size σ that can be drawn from S. Next, substituting expressions for the conditioned probabilities, we obtain

$$\Pr(A_i \mid S) = \sum_{N(s_i)=0}^{\sigma} \frac{N(s_i)}{\sigma} \frac{\binom{N_i}{N(s_i)}\binom{N-N_i}{\sigma - N(s_i)}}{\binom{N}{\sigma}}.$$

In the expression on the right $N(s_i)/\sigma$ represents the probability of A_i in the presence of a sample of size σ containing a subset s_i of elements conditioned to A_i; the product of binomial coefficients denotes the number of ways of obtaining exactly $N(s_i)$ elements conditioned to A_i in a sample of size σ, so that the ratio of this product to the number of ways of drawing a sample of size σ is the probability of obtaining the given value of $N(s_i)/\sigma$. The resulting formula will be recognized as the familiar expression for the mean of a hypergeometric distribution (Feller, 1957, p. 218), and we have the pleasingly simple outcome that the probability of a response to the stimulating situation represented by a set S is equal to the proportion of elements of S that are conditioned to the given response:

$$\Pr(A_i \mid S) = \frac{N_i}{N}. \tag{53}$$

This result may seem too intuitively obvious to have needed a proof, but it should be noted that the same theorem does not hold in general for component models with fixed sampling probabilities for the elements (cf. Estes & Suppes, 1959b).

3.4 Interpretation of Stimulus Generalization

Our approach to the problem of stimulus generalization is to represent the similarity between two stimuli by the amount of overlap between two sets of elements.[10] In the simplest experimental paradigm for exhibiting generalization we begin with two stimulus situations, represented by sets S_a and S_b, neither of which has any of its elements conditioned to a reference response A_1. Training is given by reinforcement of A_1 in the presence of S_a only until the probability of A_1 in that situation reaches some value $p_{a1} > 0$. Then test trials are given in the presence of S_b, and if p_{b1} now proves to be greater than zero we say that stimulus generalization has occurred. If the axioms of the component model are satisfied, the value of p_{b1} provides, in fact, a measure of the overlap of S_a and S_b; for, by Eq. 53, we have, immediately,

$$p_{b1} = \frac{N(S_a \cap S_b)p_{a1}}{N(S_b)},$$

where $S_a \cap S_b$ denotes the set of elements common to S_a and S_b, since the numerator of this fraction is simply the number of elements in S_b that are now conditioned to response A_1. More generally, if the proportion of elements of S_b conditioned to A_1 before the experiment were equal to g_{b1}, not necessarily zero, the probability of response A_1 to stimulus S_b after training in S_a would be given by

$$p_{b1} = \frac{N(S_a \cap S_b)p_{a1} + [N(S_b) - N(S_a \cap S_b)]g_{b1}}{N(S_b)},$$

or, with the more compact notation $N_{ab} = N(S_a \cap S_b)$, etc.,

$$p_{b1} = \frac{N_{ab}p_{a1} + (N_b - N_{ab})g_{b1}}{N_b}. \tag{54a}$$

This relation can be put in still more convenient form by letting $N_{ab}/N_b = w_{ab}$, namely,

$$p_{b1} = w_{ab}p_{a1} + (1 - w_{ab})g_{b1}.$$

This equation may be rearranged to read

$$p_{b1} = w_{ab}(p_{a1} - g_{b1}) + g_{b1}, \tag{54b}$$

and we see that the difference $(p_{a1} - g_{b1})$ between the posttraining probability of A_1 in S_a and the pretraining probability in S_b can be regarded

[10] A model similar in most essentials has been presented in Bush & Mosteller (1951b).

as the slope parameter of a linear "gradient" of generalization, in which p_{b1} is the dependent variable and the proportion of overlap between S_a and S_b is the independent variable. If we hold g_{b1} constant and let p_{a1} vary as the parameter, we generate a family of generalization gradients which have their greatest disparities at $w_{ab} = 1$ (i.e., when the test stimulus S_b is identical with S_a) and converge as the overlap between S_b and S_a decreases, until the gradients meet at $p_{b1} = g_{b1}$ when $w_{ab} = 0$. Thus the family of gradients shown in Fig. 9 illustrates the picture to be expected if a series of generalization tests is given at each of several different stages of training in S_a, or, alternatively, at several different stages of extinction following training in S_a, as was done, for example, by Guttman and Kalish (1956). The problem of "calibrating" a physical stimulus dimension to obtain a series of values that represent equal differences in the value of w_{ab} has been discussed by Carterette (1961).

The parameter w_{ab} might be regarded as an index of the similarity of S_a to S_b. In general, similarity is not a symmetrical relation, for w_{ab} is not equal to w_{ba} (w_{ab} being given by N_{ab}/N_b and the w_{ba} by N_{ab}/N_a) except in the special case $N_a = N_b$. When $N_a \neq N_b$, generalization from training with the larger set to a test with the smaller set will be greater than general-

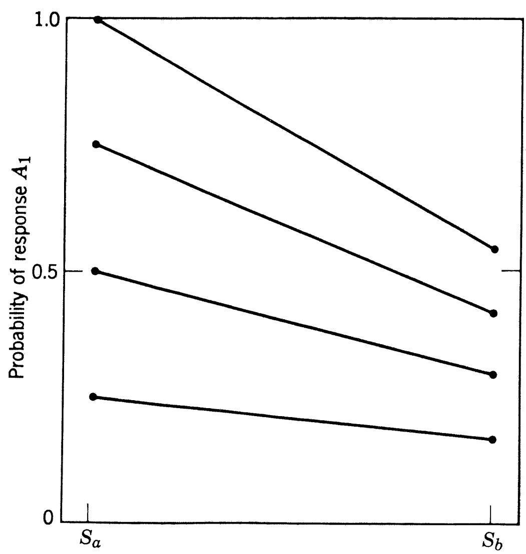

Fig. 9. Generalization from a training stimulus, S_a, to a test stimulus, S_b, at several stages of training. The parameters are $w_{ab} = 0.5$, the proportion of overlap between S_a and S_b, and $g_{b1} = 0.1$, the probability of response A_1 to S_b before training on S_a.

ization from training with the smaller set to a test with the larger set (assuming that the reinforcement given the reference response A_1 in the presence of the training set S_i establishes the same value of p_{i1} in each case before testing in S_j). We shall give no formal assumption relating size of a stimulus set to observable properties; however, it is reasonable to expect that larger sets will be associated with more intense (where the notion of intensity is applicable) or attention-getting stimuli. Thus, if S_a and S_b represent tones *a* and *b* of the same frequency but with tone *a* more intense than *b*, we should predict greater generalization if we train the reference response to a given level with *a* and test with *b* than if we train to the same level with *b* and test with *a*.

Although in the psychological literature the notion of stimulus generalization has nearly always been taken to refer to generalization along some physical continuum, such as wavelength of light or intensity of sound, it is worth noting that the set-theoretical model is not restricted to such cases. Predictions of generalization in the case of complex stimuli may be generated by first evaluating the overlap parameter w_{ab} for a given pair of situations *a* and *b* from a set of observations obtained with some particular combination of values of p_{a1} and g_{b1} and then computing theoretical values of p_{b1} for new conditions involving different levels of p_{a1} and g_{b1}. The problem of treating a simple "stimulus dimension" is of special interest, however, and we conclude our discussion of generalization by sketching one approach to this problem.[11]

We shall consider the type of stimulus dimension that Stevens (1957) has termed *substitutive* or *metathetic*, that is, one which involves the notion of a simple ordering of stimuli along a dimension without variation in intensity or magnitude. Let us denote by Z a physical dimension of this sort, for example, wavelength of visible light, which we wish to represent by a sequence of stimulus sets. First we shall outline the properties that we wish this representation to have and then spell out the assumptions of the model more rigorously.

It is part of the intuitive basis of a substitutive dimension that one moves from point to point by exchanging some of the elements of one stimulus for new ones belonging to the next. Consequently, we assume that as values of Z change by constant increments each successive stimulus set should be generated by deleting a constant number of elements from the preceding set and adding the same number of new elements to form the

[11] We follow, in most respects, the treatment given by W. K. Estes and D. L. LaBerge in unpublished notes prepared for the 1957 SSRC Summer Institute in Social Science for College Teachers of Mathematics. For an approach combining essentially the same set-theoretical model with somewhat different learning assumptions, the reader is referred to Restle (1961).

next set; but, to ensure that the organism's behavior can reflect the ordering of stimuli along the Z-scale without ambiguity, we need also to assume that once an element is deleted as we go along the Z-scale it must not reappear in the set corresponding to any higher Z-value. Further, in view of the abundant empirical evidence that generalization declines in an orderly fashion as the distance between two stimuli on such a dimension increases, we must assume that (at least up to the point at which sets corresponding to larger differences in Z are disjoint) the overlap between two stimulus sets is directly related to the interval between the corresponding stimuli on the Z-scale. These properties, taken together, enable us to establish an intuitively reasonable correspondence between characteristics of a sequence of stimulus sets and the empirical notion of generalization along a dimension.

These ideas are incorporated more formally in the following set of axioms. The basis for these axioms is a stimulus dimension Z, which may be either continuous or discontinuous, a collection S_* of stimulus sets, and a function $x(Z)$ with a finite number of consecutive integers in its range. The mapping of the set (x) of scaled stimulus values onto the subsets S_i of S_* must satisfy the following axioms:

Generalization Axioms

G1. *For all* $i \leqslant j \leqslant k$ *in* (x), $S_i \cap S_k \subseteq S_j$.

G2. *For all* $i \leqslant j \leqslant k$ *in* (x), *if* $S_i \cap S_k \neq \emptyset$, *where* $\emptyset$ *is the null set, then* $S_j \subseteq (S_i \cup S_k)$.

G3. *For all* $h < i$, $j < k$ *in* (x), *if* $i - h = k - j$, *then* $N_{hi} = N_{jk}$; *and for all* i *in* (x), $N_{ii} = N$.

The set (x) may simply be a set of Z scale values or it may be a set of Z-values rescaled by some transformation. The reasons for introducing (x) are twofold. First, for mathematical simplicity we find it advisable to restrict ourselves, at least for present purposes, to a finite set of Z-values and therefore to a finite collection of stimulus sets. Second, there is no reason to believe that equal distances along physical dimensions will in general correspond to equal overlaps between stimulus sets. All that is required, however, to make the theory workable is that for any given physical dimension, wavelength of light, frequency of a tone, or whatever, we can find experimentally a transformation x such that equal distances on the x-scale do correspond to equal overlaps.

Axiom G1 states that if an element belongs to any two sets it also belongs to all sets that fall between these two sets on the x-scale. Axiom G2 states that if two sets have any common elements then all of the elements of any set falling between them belong to one or the other (or both) of the given sets; this property ensures that the elements drop out of the sets in order as we move along the dimension. Axiom G3 describes the property that distinguishes a simple substitutive dimension from an additive, or intensity (in Stevens' terminology, *prothetic*), dimension. It should be noted that only if the number of values in the range of $x(Z)$ is no greater than $N(S_*) - N + 1$ can Axiom G3 be satisfied. This restriction is necessary in order to obtain a one-to-one mapping of the x-values into the subsets S_i of S_*.

One advantage in having the axioms set forth explicitly is that it then becomes relatively easy to design experiments bearing on various aspects of the model. Thus, to obtain evidence concerning the empirical tenability of Axiom G1, we might choose a response A_1 and a set (x) of stimuli, including a pair i and k such that $\Pr(A_1 \mid i) = \Pr(A_1 \mid k) = 0$, then train subjects with stimulus i only until $\Pr(A_1 \mid i) = 1$, and finally test with stimulus k. If $\Pr(A_1 \mid k)$ is found to be greater than zero, it must be concluded, in terms of the model, that $S_i \cap S_k \neq \emptyset$; that is, the sets corresponding to i and k have some elements in common. Given

$$\Pr(A_1 \mid k) > 0,$$

it must be predicted that for every stimulus j in (x), such that $i < j < k$, $\Pr(A_1 \mid j) \geqslant \Pr(A_1 \mid k)$. Axiom G1 ensures that all of the elements of S_k which are now conditioned to A_1 by virtue of belonging also to S_i must be included in S_j, possibly augmented by other elements of S_i which are not in S_k.

To deal similarly with Axiom G2, we proceed in the same way to locate two members i and k of a set (x) such that $S_i \cap S_k \neq \emptyset$. Then we train subjects on both stimulus i and stimulus k until $\Pr(A_1 \mid i) = \Pr(A_1 \mid k) = 1$, response A_1 being one that before this training had probability of less than unity to all stimuli in (x). Now, by G2, if any stimulus j falls between i and k, the set S_j must be contained entirely in the union $S_i \cup S_k$; consequently, we must predict that we will now find $\Pr(A_1 \mid j) = 1$ for any stimulus j such that $i \leqslant j \leqslant k$.

To evaluate Axiom G3 empirically, we require four stimuli $h < i, j < k$ such that $i - h = k - j$. If the four stimuli are all different, we can simply train subjects on h and test generalization to i, then train subjects to an equal degree on j and test generalization to k. If the amount of generalization, as measured by the probability of the test response, is the same in the two cases, then the axiom is supported. In the special case in which $h = i$ and $j = k$ we would be testing the assertion that the sets associated with different values of x are of equal size. To accomplish this test, we need only take any two neighboring values of x, say i and j, train subjects to some criterion on i and test on j, then reverse the procedure by training (different) subjects to the same criterion on j and testing on i. If the axiom is satisfied, the amount of generalization should be the same in both directions.

Once we have introduced the notion of a dimension, it is natural to inquire whether the parameter that represents the degree of communality between pairs of stimulus sets might not be related in some simple way to a measure of distance along the dimension. With one qualification, which we mention later, the quantity $d_{ij} = 1 - w_{ij}$ could serve as a suitable measure of the distance between stimuli i and j. We can check to see whether the familiar axioms for a metric are satisfied. These axioms are

1. $d_{ij} = 0$ if and only if $i = j$,
2. $d_{ij} \geqslant 0$,
3. $d_{ij} = d_{ji}$,
4. $d_{ij} + d_{jk} \geqslant d_{ik}$,

where it is understood that i, j, and k are any members of the set (x) associated with a given dimension. The first three obviously hold, but the fourth requires a bit of analysis. To carry out a proof, we use the notation N_{ij} for the number of elements common to S_i and S_j, $N_{ij\bar{k}}$ for the number of elements in both S_i and S_j but not in S_k, and so on. The difference between the two sides of the inequality we wish to establish can be expanded in terms of this notation:

$$d_{ij} + d_{jk} - d_{ik} = \left(1 - \frac{N_{ij}}{N}\right) + \left(1 - \frac{N_{jk}}{N}\right) - \left(1 - \frac{N_{ik}}{N}\right)$$

$$= \frac{1}{N}(N - N_{ij} - N_{jk} + N_{ik})$$

$$= \frac{1}{N}(N_{ijk} + N_{ij\bar{k}} + N_{\bar{i}jk} + N_{\bar{i}j\bar{k}} - N_{ijk} - N_{ij\bar{k}} - N_{ijk}$$

$$- N_{\bar{i}jk} + N_{ijk} + N_{i\bar{j}k})$$

$$= \frac{1}{N}(N_{\bar{i}j\bar{k}} + N_{i\bar{j}k}).$$

The last expression on the right is nonnegative, which establishes the desired inequality. To find the restrictions under which d is additive, let us assume that stimuli i, j, and k fall in the order $i < j < k$ on the dimension. Then, by Axiom G1, we know that $N_{i\bar{j}k} = 0$. However it is only in the special cases in which S_i and S_k are either overlapping or adjacent that $N_{\bar{i}j\bar{k}} = 0$

and, therefore, that $d_{ij} + d_{jk} = d_{ik}$. It is possible to define an additive distance measure that is not subject to this restriction, but such extensions raise new problems and we are not able to pursue them here.

In concluding this section, we should like to emphasize one difference between the model for generalization sketched here and some of those already familiar in the literature (see, e.g., Spence, 1936; Hull, 1943). We do not postulate a particular form for generalization of response strength or excitatory tendency. Rather, we introduce certain assumptions about the properties of the set of stimuli associated with a sensory dimension; then we take these together with learning assumptions and information about reinforcement schedules as a basis for deriving theoretical gradients of generalization for particular types of experiments. Under the special conditions assumed in the example we have considered, the theory predicts that a family of linear gradients with simple properties will be observed when response probability is plotted as a function of distance from the point of reinforcement. Predictions of this sort may reasonably be tested by means of experiments in which suitable measures are taken to meet the conditions assumed in the derivations (see, e.g., Carterette, 1961); but, to deal with experiments involving different training conditions or response measures other than relative frequencies, further theoretical analysis is called for, and we must be prepared to find substantial differences in the phenotypic properties of generalization gradients derived from the same basic theory for different experimental situations.

4. COMPONENT AND LINEAR MODELS FOR SIMPLE LEARNING

In this section we combine, in a sense, the theories discussed in the preceding sections. Until now it was convenient for expositional purposes to treat the problems of learning and generalization separately. We first considered a type of learning model in which the different possible samples of stimulation from trial to trial were assumed to be entirely distinct and then turned to an analysis of generalization, or transfer, effects that could be measured on an isolated test trial following a series of learning trials. Prediction of these transfer effects depended on information concerning the state of the stimulus population just before the test trial but did not depend on information about the course of learning over preceding training trials. However, in many (perhaps most) learning situations it is not reasonable to assume that the samples, or patterns, of stimulation affecting the organism on different trials of a series are entirely disjoint; rather, they must overlap to various intermediate degrees, thus generating transfer effects throughout the learning series. In the "component models" of stimulus sampling theory one simply takes the learning assumptions of the pattern model (Sec. 2) together with the sampling axioms and response rule of the generalization model (Sec. 3) to generate an account of learning for this more general case.

4.1 Component Models with Fixed Sample Size

As indicated earlier, the analysis of a simple learning experiment in terms of a component model is based on the representation of the stimulus as a set S of N stimulus elements from which the subject draws a sample on each trial. At any time, each element in the set S is conditioned to exactly one of the r response alternatives $A_1, \ldots, A_r$; by the response axiom of Sec. 3.1 the probability of a response is equal to the proportion of elements in the trial sample conditioned to that response. At the termination of a trial, if reinforcing event E_i $(i \neq 0)$ occurs, then with probability c all elements in the trial sample become conditioned to response A_i. If E_0 occurs, the conditioned status of elements in the sample does not change. The conditioning parameter c plays the same role here as in the pattern model. It should be noted that in the early literature of stimulus sampling theory this parameter was usually assumed to be equal to unity.

Two general types of component models can be distinguished. For the *fixed-sample-size* model we assume that the sample size is a fixed number s throughout any given experiment. For the *independent-sampling* model we assume that the elements of the stimulus set S are sampled independently on each trial, each element having some fixed probability θ of being drawn. In this section we discuss the fixed-sample-size model and consider the case in which all possible samples of size s are sampled with equal probability.

FORMULATION FOR *RTT* EXPERIMENTS. To illustrate the model, we first consider an experimental procedure in which a particular stimulus item is given a single reinforced trial, followed by two consecutive non-reinforced test trials. The design may be conveniently symbolized RT_1T_2. Procedures and results for a number of experiments using an RTT design have been reported elsewhere (Estes, 1960a; Estes, Hopkins, & Crothers, 1960; Estes, 1961b; Crothers, 1961). For simplicity, suppose we select a situation in which the probability of a correct response is zero before the first reinforcement (and in which the likelihood of a subject's obtaining correct responses by guessing is negligible on all trials). In terms of the fixed-sample-size model we can readily generate predictions for the probabilities p_{ij} of various combinations of response i on T_1 and response j

on T_2. If $i, j = 0$ denote correct responses and $i, j = 1$ denote errors, then

$$
\begin{aligned}
p_{00} &= c\left(\frac{s}{N}\right)^2 \\
p_{01} &= c\left(\frac{s}{N}\right)\left(1 - \frac{s}{N}\right) \\
p_{10} &= c\left(1 - \frac{s}{N}\right)\frac{s}{N} \\
p_{11} &= 1 - c + c\left(1 - \frac{s}{N}\right)^2.
\end{aligned}
\tag{55}
$$

To obtain the first result, we note that the correct response can occur on either trial only if conditioning occurs on the reinforced trial, which has probability c. On occasions when conditioning occurs, the whole sample of s elements becomes conditioned to the correct response and the probability of this response on each of the test trials is s/N. On occasions when conditioning does not occur on the reinforced trial, probability of a correct response remains at zero over both test trials. Note that when $s = N = 1$ this model is equivalent to the one-element model discussed in Sec. 1.1. If more than one reinforcement is given prior to T_1, the predictions are essentially unchanged. In general, for k preceding reinforcements, the expected proportion of elements conditioned to the correct response (i.e., the probability of a correct response) at the time of the first test is

$$p_0 = 1 - \left(1 - \frac{cs}{N}\right)^k,$$

and the probability of correct responses on both T_1 and T_2 is given by

$$p_{00} = \sum_{i=1}^{k} \binom{k}{i} c^i(1 - c)^{k-i}\left[1 - \left(1 - \frac{s}{N}\right)^i\right]^2.$$

To obtain this last expression, we note that a subject for whom i of the k reinforcements have been effective will have probability $\{1 - [1 - (s/N)]^i\}$ of making a correct response on each test, and the probability that exactly i reinforcements will be effective is $\binom{k}{i} c^i(1 - c)^{k-i}$. Similarly,

$$p_{10} = p_{01} = \sum_{i=1}^{k} \binom{k}{i} c^i(1 - c)^{k-i}\left[1 - \left(1 - \frac{s}{N}\right)^i\right]\left(1 - \frac{s}{N}\right)^i,$$

and

$$p_{11} = (1 - c)^k + \sum_{i=1}^{k} \binom{k}{i} c^i(1 - c)^{k-i}\left(1 - \frac{s}{N}\right)^{2i}.$$

If $s = N$, these expressions reduce to

$$p_{00} = 1 - (1 - c)^k$$
$$p_{10} = p_{01} = 0$$
$$p_{11} = (1 - c)^k.$$

This special case appears well suited to the interpretation of data obtained by G. H. Bower (personal communication) from a study in which the T_1T_2 procedure was applied following various numbers of presentations of word-word paired-associates. For 32 subjects, each tested on 10 items, Bower reports observed proportions of $p_{00} = 0.894$, $p_{10} = p_{01} = 0.003$, and $p_{11} = 0.100$.

When applied to other *RTT* experiments, this model has, however, not yielded consistently accurate predictions. The difficulty apparently stems from the fact that our assumptions do not take account of the retention loss that is usually observed from T_1 to T_2 (see, e.g., Estes, 1961b). An extension of the model that is capable of handling retention decrement as well as the acquisition process is discussed in Sec. 4.2 below.

For *RTT* experiments, in which the probability of successful guessing is not negligible (as in paired-associate tasks involving a fixed list of responses which are known to the subject from the start), some additional considerations arise. Perhaps the most natural extension of the preceding treatment is to assume that the subject will start the experiment with a proportion $1/r$ of the elements of a given set S_i connected to the correct response and a proportion $[1 - (1/r)]$ connected to incorrect responses, r being the number of alternative responses. Then, for a fixed-sample-size model, the probability p_0 of a correct response to a given item on the first test trial after a single reinforcement is

$$p_0 = (1 - c)\frac{1}{r} + c\left[\frac{s + (N - s)/r}{N}\right]$$
$$= \left(1 - \frac{cs}{N}\right)\frac{1}{r} + \frac{cs}{N},$$

the bracketed quantity being the proportion of elements connected to the correct response in the event that the reinforcement is effective. The probabilities of various combinations of correct and incorrect responses on the two test trials are given by

$$p_{00} = (1 - c)\frac{1}{r^2} + c\phi^2$$
$$p_{10} = p_{01} = (1 - c)\frac{1}{r}\left(1 - \frac{1}{r}\right) + c\phi(1 - \phi) \qquad (56)$$
$$p_{11} = (1 - c)\left(1 - \frac{1}{r}\right)^2 + c(1 - \phi)^2,$$

where

$$\phi = \frac{s}{N} + \left(1 - \frac{s}{N}\right)\frac{1}{r}.$$

An alternative approach to the type of experiment in which the subject guesses on unlearned items is to assume that initially all elements are neutral, that is, are connected neither to correct nor to incorrect responses. In the presence of a sample containing only neutral elements the subject guesses, with probability $1/r$ of being correct. If the sample contains any conditioned elements, then the proportion of conditioned elements in the sample connected to the correct response determines its probability (e.g., if the sample contains nine elements, three conditioned to the correct response, two conditioned to an incorrect response, and four unconditioned, then the probability of a correct response is simply 3/5). These assumptions seem in some respects more intuitively satisfactory than those we have considered. Perhaps the most important difference with respect to empirical implications lies in the fact that with the latter set of assumptions exposure time on test trials must be taken into account. If the stimulus exposure time is just long enough to permit a response (in terms of the theory, just long enough to permit the subject to draw a single sample of stimulus elements), then the probabilities of correct and incorrect response combinations on T_1 and T_2 are

$$\begin{aligned} p_{00} &= (1 - c)\frac{1}{r^2} + c\phi'^2, \\ p_{10} = p_{01} &= (1 - c)\frac{1}{r}\left(1 - \frac{1}{r}\right) + c\phi'(1 - \phi'), \\ p_{11} &= (1 - c)\left(1 - \frac{1}{r}\right)^2 + c(1 - \phi')^2, \end{aligned} \tag{57}$$

where

$$\phi' = 1 - \left(1 - \frac{1}{r}\right)\frac{\binom{N-s}{s}}{\binom{N}{s}}.$$

The factor $\binom{N-s}{s}\Big/\binom{N}{s}$ is the probability that the subject will draw a sample containing none of the s elements that became conditioned on the reinforced trial; therefore $1 - \phi'$ represents the probability that a subject for whom the reinforced trial was effective nevertheless draws a sample containing no conditioned elements and makes an incorrect guess, whereas ϕ' is the probability that such a subject will make a correct response on either test trial.

The two sets of equations (56 and 57) are formally identical and thus cannot be distinguished in application to *RTT* data. Like Eq. 55, they have the limitation of not allowing adequately for the retention loss usually observed (see, e.g., Estes, Hopkins, & Crothers, 1960); we return to this point in Sec. 4.2.

If exposure time is long enough on the test trials, then we assume that the subject continues to draw successive random samples from S and makes a response only when he finally draws a sample containing at least one conditioned element. Thus in cases in which the reinforcement has been effective on a previous trial (so that S contains a subset of s conditioned elements) the subject will eventually draw a sample containing one or more conditioned elements and will respond on the basis of these elements, thereby making a correct response with probability 1. Therefore, for the case of unlimited exposure time, $\phi' = 1$ and Eq. 57 reduces to

$$\begin{aligned} p_{00} &= (1 - c)\frac{1}{r^2} + c, \\ p_{10} = p_{01} &= (1 - c)\frac{1}{r}\left(1 - \frac{1}{r}\right), \\ p_{11} &= (1 - c)\left(1 - \frac{1}{r}\right)^2, \end{aligned} \tag{58}$$

which are identical with the corresponding equations for the one-element model of Sec. 1.2.

GENERAL FORMULATION. We turn now to the problem of deriving from the fixed-sample-size model predictions concerning the course of learning over an experiment consisting of a sequence of trials run under some prescribed reinforcement schedule. We shall limit consideration to the case in which each element in S is conditioned to exactly one of the two response alternatives, A_1 or A_2, so that there are $N + 1$ conditioning states. Again, we let C_i $(i = 0, \ldots, N)$ denote the state in which i elements of the set S are conditioned to A_1 and $N - i$ to A_2. As in the pattern model, the transition probabilities among conditioning states are functions of the reinforcement schedules and the set-theoretical parameters c, s, and N. Following our approach in Sec. 2.1, we restrict the analysis to cases in which the probability of reinforcement depends at most on the response on the given trial; we thereby guarantee that all elements in the transition matrix for conditioning states are constant over trials. Thus the sequence of conditioning states can again be conceived as a Markov chain.

Transition Probabilities. Let $s_{i,n}$ denote the event of drawing a sample on trial n with i elements conditioned to A_1 and $s - i$ conditioned to A_2. Then the probability of a one-step transition from state C_j to state C_{j+v} is given by

$$q_{j,j+v} = c \frac{\binom{N-j}{v}\binom{j}{s-v}}{\binom{N}{s}} \Pr(E_1 \mid s_{s-v}C_j), \tag{59a}$$

where $\Pr(E_1 \mid s_{s-v}C_j)$ is the probability of an E_1-event, given conditioning state C_j and a sample with v elements conditioned to A_2. To obtain Eq. 59*a*, we note that an E_1 must occur and that the subject must sample exactly v elements from the $N - j$ elements not already conditioned to A_1; the probability of the latter event is the number of ways of drawing samples with v elements conditioned to A_2 divided by the total number of ways of drawing samples of size s. Similarly

$$q_{j,j-v} = c \frac{\binom{N-j}{s-v}\binom{j}{v}}{\binom{N}{s}} \Pr(E_2 \mid s_v C_j) \tag{59b}$$

and

$$q_{j,j} = 1 - c + c\left[\frac{\binom{j}{s}}{\binom{N}{s}} \Pr(E_1 \mid s_s C_j) + \frac{\binom{N-j}{s}}{\binom{N}{s}} \Pr(E_2 \mid s_0 C_j) + \Pr(E_0 \mid C_j)\right]. \tag{59c}$$

Although it is an obvious conclusion, it is important for the reader to realize that the pattern model discussed in Sec. 2 is identical to the fixed-sample-size model when $s = 1$. This correspondence between the two models is indicated by the fact that Eqs. 59*a*, *b*, *c* reduce to Eq. 23*a*, *b*, *c* when we let $s = 1$.

For the simple noncontingent schedule in which only the two events E_1 and E_2 occur (with probabilities π and $1 - \pi$, respectively) Eqs. 59*a*, *b*, *c* simplify to

$$q_{j,j+\nu} = c\pi \frac{\binom{N-j}{\nu}\binom{j}{s-\nu}}{\binom{N}{s}}, \tag{60a}$$

$$q_{j,j-\nu} = c(1-\pi) \frac{\binom{N-j}{s-\nu}\binom{j}{\nu}}{\binom{N}{s}}, \tag{60b}$$

$$q_{j,j} = 1 - c + c\left[\pi \frac{\binom{j}{s}}{\binom{N}{s}} + (1-\pi)\frac{\binom{N-j}{s}}{\binom{N}{s}}\right]. \tag{60c}$$

It is apparent that state C_N is an absorbing state when $\pi = 1$ and that C_0 is an absorbing state when $\pi = 0$. Otherwise, all states are ergodic.

Mean Learning Curve. Following the same techniques used in connection with Eq. 27, we obtain for the component model in the simple, noncontingent case

$$\Pr(A_{1,n}) = \pi - [\pi - \Pr(A_{1,1})]\left(1 - \frac{cs}{N}\right)^{n-1}. \tag{61}$$

This mean learning function traces out a smooth growth curve that can take any value between 0 and 1 on trial n if parameters are selected appropriately. However, it is important to note that for a given realization of the experiment the actual response probabilities for individual subjects (as opposed to expectations) can only take on the values $0, 1/N, 2/N, \ldots, (N-1)/N, 1$; that is, the values associated with the conditioning states. This stepwise aspect of the process is particularly important when one attempts to distinguish between this model and models that assume gradual continuous increments in the strength or probability of a response over time (Hull, 1943; Bush & Mosteller, 1955; Estes & Suppes, 1959a).

To illustrate this point, we consider an experiment on avoidance learning reported by Theios (1963). Fifty rats were used as subjects. The apparatus was a modified Miller-Mowrer electric-shock box, and the animal was always placed in the black compartment. Shortly thereafter a buzzer and light came on as the door between the compartments was opened. The correct response (A_1) was to run into the other compartment within 3 seconds. If A_1 did not occur, the subject was given a high intensity shock until it escaped into the other compartment. After 20 seconds the subject was returned to the black compartment, and another trial was given.

Each rat was run until it met a criterion of 20 consecutive successful avoidance responses.

Theios analyzed the situation in terms of a component model in which $N = 2$ and $s = 1$. Further, he assumed that $\Pr(A_{1,1}) = 0$, hence on trial 1 the subject is in conditioning state C_0. Employing Eq. 60 with $\pi = 1$, $N = 2$, and $s = 1$, we obtain the following transition matrix:

$$\begin{array}{c} \\ C_2 \\ C_1 \\ C_0 \end{array} \begin{array}{c} \begin{array}{ccc} C_2 & C_1 & C_0 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{c}{2} & 1 - \dfrac{c}{2} & 0 \\ 0 & c & 1 - c \end{bmatrix} \end{array}.$$

The expected probability of an A_1-response on trial n is readily obtained by specialization of Eq. 61,

$$\Pr(A_{1,n}) = 1 - \left(1 - \frac{c}{2}\right)^{n-1}.$$

Applying this model, Theios estimated $c = 0.43$ and provided an impressive account of such statistics as total errors, the mean learning curve, trial number of last error, autocorrelation of errors with lags of 1, 2, 3, and 4 trials, mean number of runs, probability of no reversals, and many others. However, for our immediate purposes we are interested in only one feature of his data; namely, whether the underlying response probabilities are actually fixed at 0, $\frac{1}{2}$, and 1, as specified by the model. First we note that it is not possible to establish the exact trial on which the subject moves from C_0 to C_1 or from C_1 to C_2. Nevertheless, if there are some trials between the first success (A_1-response) and the last error (A_2-response), we can be sure that the subject is in state C_1 on these trials, for, if the subject has made one success, at least one of the two stimulus elements is conditioned to the A_1-response; if on a later trial the subject makes an error, then, up to that trial, at least one of the elements is not conditioned to the A_1-response. Since deconditioning does not occur in the present model, the subject must be in conditioning state C_1. Thus, according to the model, the sequence of responses after the first success and before the last error should form a sequence of Bernoulli trials with constant probability $p = q = \frac{1}{2}$ of an A_1-response. Theios has applied several statistical tests to check this hypothesis and none suggests that the assumption is incorrect. For example, the response sequences for the trials between the first success and last error were divided into blocks of four trials and the number of A_1-responses in each block was counted. The obtained frequencies for 0, 1, 2, 3, and 4 successes were 2, 12, 17, 15, and 4, respectively;

the predicted binomial frequencies were 3.1, 12.5, 18.5, 12.5, and 3.1. The correspondence between predicted and observed frequencies is excellent, as indicated by a χ^2 goodness-of-fit test that yielded a value of 1.47 with 4 degrees of freedom.

Theios has applied the same analysis to data from an experiment by Solomon and Wynne (1953), in which dogs were required to learn an avoidance response. The findings with regard to the binomial property on trials after the first success and before the last error are in agreement with his own data but suggest that the binomial parameter is other than $\frac{1}{2}$. From a stimulus sampling viewpoint this observation would suggest that the two elements are not sampled with equal probabilities. For a detailed discussion of this Bernoulli stepwise aspect of certain stimulus sampling models, related statistical tests, and a review of relevant experimental data the reader is referred to Suppes & Ginsberg (1963).

The mean learning curve for the fixed sample size model given by Eq. 60 is identical to the corresponding equation for the pattern model with the sampling ratio cs/N taking the role of c/N. However, we need not look far to find a difference in the predictions generated by the two models. If we define $\alpha_{2,n}$ as in Eq. 29, that is,

$$\alpha_{2,n} = \sum_{i=0}^{N} \frac{i^2}{N^2} \Pr(C_{i,n}),$$

then by carrying out the summation, using the same methods as in the case of Eq. 27, we obtain

$$\alpha_{2,n} = \left[1 - \frac{2cs}{N} + \frac{cs(s-1)}{N(N-1)}\right]\alpha_{2,n-1} + \frac{c}{N}\left[\frac{s}{N} - \frac{s(s-1)}{N(N-1)}\right]\alpha_{1,n-1} + 2c\pi\left(\frac{s}{N} - \frac{s^2}{N^2}\right)\alpha_{1,n-1} + \frac{c\pi s^2}{N^2}. \qquad (62)$$

The asymptotic variance of the response probabilities for the component model is simply

$$\sigma_\infty^{\ 2} = \alpha_{2,\infty} - [\Pr(A_{1,\infty})]^2.$$

Letting $\alpha_{2,n} = \alpha_{2,n-1} = \alpha_{2,\infty}$, noting that $\Pr(A_{1,\infty}) = \pi$ and carrying out the appropriate computations, we obtain

$$\sigma_\infty^{\ 2} = \frac{\pi(1-\pi)}{N}\left[\frac{N + (N-2)s}{2N - s - 1}\right]. \qquad (63)$$

This asymptotic variance of the response probabilities depends in relatively simple ways on s and N. If we hold N fixed and differentiate with respect to s, we find that σ_∞^2 increases monotonically with s; in particular, then, this variance for a fixed sample size model with $s > 1$ is larger than that of the pattern model with the same number of elements. If we hold the sampling ratio s/N fixed and take the partial derivative with respect to N, we find σ_∞^2 to be a decreasing function of N. In the limit, if $N \to \infty$ in such a way that $s/N = \theta$ remains constant, then

$$\sigma_\infty^2 \longrightarrow \pi(1 - \pi)\frac{\theta}{2 - \theta}, \tag{64}$$

which, we shall see later, is the variance for the linear model (Estes & Suppes, 1959a). In contrast, for the pattern model the variance of the p-values approaches 0 as N becomes large. We return to comparisons between the two models in Sec. 4.3.

Sequential Predictions. We now examine some sequential statistics for the fixed-sample-size model which later will help to clarify relationships among the various stimulus sampling models. As in previous cases (e.g., Eq. 31*a*), we give results only for the noncontingent case in which $\Pr(E_{0,n}) = 0$ and $r = 2$.

Consider, first, $\Pr(A_{1,n+1} \mid E_{1,n})$. By taking account of the conditioning states on trial $n + 1$ and trial n and also the sample on trial n we may write

$$\Pr(A_{1,n+1} \mid E_{1,n}) = \frac{1}{\Pr(E_{1,n})} \sum_{i,j,k} \Pr(A_{1,n+1}C_{j,n+1}E_{1,n}s_{i,n}C_{k,n}),$$

where, as before, $s_{i,n}$ denotes the event of drawing a sample on trial n with i elements conditioned to A_1 and $s - i$ conditioned to A_2. Conditionalizing, with our learning axioms in mind, we obtain

$$\begin{aligned}\Pr(A_{1,n+1} \mid E_{1,n}) &= \frac{1}{\Pr(E_{1,n})} \sum_{i,j,k} \Pr(A_{1,n+1} \mid C_{j,n+1}) \Pr(C_{j,n+1} \mid E_{1,n}s_{i,n}C_{k,n}) \\ &\quad \cdot \Pr(E_{1,n} \mid s_{i,n}C_{k,n}) \Pr(s_{i,n} \mid C_{k,n}) \Pr(C_{k,n}).\end{aligned}$$

But for our reinforcement procedures $\Pr(E_{1,n}) = \Pr(E_{1,n} \mid s_{i,n}C_{k,n})$. Further

$$\Pr(C_{j,n+1} \mid E_{1,n}s_{i,n}C_{k,n}) = \begin{cases} c & \text{if } j = k + s - i, \\ 1 - c & \text{if } j = k, \\ 0 & \text{otherwise;} \end{cases}$$

that is, the $s - i$ elements in the sample originally conditioned to A_2 now become conditioned to A_1 with probability c, hence a move from state C_k to C_{k+s-i} occurs. Also, as noted with regard to Eq. 59,

$$\Pr(s_{i,n} \mid C_{k,n}) = \frac{\binom{k}{i}\binom{N-k}{s-i}}{\binom{N}{s}}.$$

Substitution of these results in our last expression for $\Pr(A_{1,n+1} \mid E_{1,n})$ yields

$$\Pr(A_{1,n+1} \mid E_{1,n}) = \sum_{i,k}\left[c\,\frac{k+s-i}{N} + (1-c)\frac{k}{N}\right]\frac{\binom{k}{i}\binom{N-k}{s-i}}{\binom{N}{s}}\Pr(C_{k,n}).$$

We now need the fact that the first raw moment of the hypergeometric distribution is

$$\sum_{i=0}^{k} i\,\frac{\binom{k}{i}\binom{N-k}{s-i}}{\binom{N}{s}} = \frac{sk}{N},$$

permitting the simplification

$$\Pr(A_{1,n+1} \mid E_{1,n}) = \sum_{k}\left[\frac{cs}{N} + \frac{k}{N}\left(1 - \frac{cs}{N}\right)\right]\Pr(C_{k,n});$$

but, by definition,

$$\Pr(A_{1\,n}) = \sum_{k}\frac{k}{N}\Pr(C_{k,n}),$$

whence

$$\Pr(A_{1,n+1} \mid E_{1,n}) = \left(1 - \frac{cs}{N}\right)\Pr(A_{1,n}) + \frac{cs}{N}. \tag{65a}$$

By the same method of proof we may show that

$$\Pr(A_{1,n+1} \mid E_{2,n}) = \left(1 - \frac{cs}{N}\right)\Pr(A_{1,n}) \tag{65b}$$

Finally, for comparison with other models, we present the expressions for

$\Pr(A_{k,n+1}E_{j,n}A_{i,n})$. Derivations of these probabilities are based on the same methods used in connection with Eq. 61a.

$$\Pr(A_{1,n+1}E_{1,n}A_{1,n}) = \pi\left\{\left[1 - \frac{c(s-1)}{N-1}\right]\alpha_{2,n} + \frac{c(s-1)}{N-1}\,\alpha_{1,n}\right\}. \tag{66a}$$

$$\Pr(A_{1,n+1}E_{1,n}A_{2,n}) = \pi\left\{\frac{cs}{N}(1-\alpha_{1,n}) + \left[1 - \frac{c(s-1)}{N-1}\right](\alpha_{1,n} - \alpha_{2,n})\right\}. \tag{66b}$$

$$\Pr(A_{1,n+1}E_{2,n}A_{1,n}) = (1-\pi)\left\{\left[1 - \frac{c(s-1)}{N-1}\right]\alpha_{2,n} - \left[\frac{cs}{N} - \frac{c(s-1)}{N-1}\right]\alpha_{1,n}\right\}. \tag{66c}$$

$$\Pr(A_{1,n+1}E_{2,n}A_{2,n}) = (1-\pi)\left[1 - \frac{c(s-1)}{N-1}\right](\alpha_{1,n} - \alpha_{2,n}). \tag{66d}$$

$$\Pr(A_{2,n+1}E_{1,n}A_{1,n}) = \pi\left[1 - \frac{c(s-1)}{N-1}\right](\alpha_{1,n} - \alpha_{2,n}). \tag{66e}$$

$$\Pr(A_{2,n+1}E_{1,n}A_{2,n}) = \pi\left\{\left(1 - \frac{cs}{N}\right)(1-\alpha_{1,n}) - \left[1 - \frac{c(s-1)}{N-1}\right](\alpha_{1,n} - \alpha_{2,n})\right\}. \tag{66f}$$

$$\Pr(A_{2,n+1}E_{2,n}A_{1,n}) = (1-\pi)\left\{\left[1 + \frac{cs}{N} - \frac{c(s-1)}{N-1}\right]\alpha_{1,n} - \left[1 - \frac{c(s-1)}{N-1}\right]\alpha_{2,n}\right\}. \tag{66g}$$

$$\Pr(A_{2,n+1}E_{2,n}A_{2,n}) = (1-\pi)\left\{1 - \alpha_{1,n} - \left[1 - \frac{c(s-1)}{N-1}\right](\alpha_{1,n} - \alpha_{2,n})\right\}. \tag{66h}$$

Application of these equations to the corresponding set of trigram proportions for a preasymptotic trial block is not particularly rewarding. The difficulty is that certain combinations of parameters, for example, $\{1 - [c(s-1)/N-1]\}(\alpha_{1,n} - \alpha_{2,n})$ and cs/N, behave as units; consequently, the basic parameters c, s, and N cannot be estimated individually and, as a result, the predictions available from the simpler N-element pattern model via Eq. 32 cannot be improved upon by use of Eq. 66. For asymptotic data the situation is somewhat different. By substituting the limiting values for $\alpha_{1,n}$ and $\alpha_{2,n}$ in Eq. 66, that is, $\alpha_1 = \pi$ and from Eq. 63

$$\alpha_2 = \sigma_\infty{}^2 + \pi^2 = \frac{\pi(1-\pi)}{N}\left[\frac{N + (N-2)s}{2N - s - 1}\right] + \pi^2$$

$$= \frac{\pi[N - 2s + Ns + 2\pi(N-s)(N-1)]}{N(2N - s - 1)},$$

we can express the trigram probabilities $\Pr(A_{k,\infty}E_{j,\infty}A_{i,\infty})$ in terms of the basic parameters of the model. The resulting expressions are somewhat cumbersome, however, and we shall not pursue this line of analysis here.

4.2 Component Models with Stimulus Fluctuation

In Sec. 4.1, as in most of the literature on stimulus sampling models for learning, we restricted attention to the special case in which the stimulation effective on successive trials of an experiment may be considered to represent independent random samples from the population of elements available under the given experimental conditions. More generally, we would expect that the independence of successive samples would depend on the interval between trials. The concept of stimulus sampling in the model corresponds to the process of stimulation in the empirical situation. Thus sampling and resampling from a stimulus population must take time; and, if the interval between trials is sufficiently short, there will not be time to draw a completely new sample. We should expect the correlation, or degree of overlap, between successive stimulus samples to vary inversely with the intertrial interval, running from perfect overlap in the limiting case (not necessarily empirically realizable) of a zero interval to independence at sufficiently long intervals. These notions have been embodied in the *stimulus fluctuation model* (Estes, 1955a, 1955b, 1959a). In this section we shall develop the assumption of stimulus fluctuation in connection with fixed-sample-size models; consequently, the expressions derived will differ in minor respects from those of the earlier presentations (cited above) that were not restricted to the case of fixed sample size.

ASSUMPTIONS AND DERIVATION OF RETENTION CURVES. Following the convention of previous articles on stimulus fluctuation models, we denote by S^* the set of stimulus elements potentially available for sampling under a given set of experimental conditions, by S the subset of elements available for sampling at any given time, and by S' the subset of elements that are temporarily unavailable (so that $S^* = S \cup S'$). The trial sample s is in turn a subset of S; however, in this presentation we assume for simplicity that all of the temporarily available elements are sampled on each trial (i.e., $S = s$). We denote by N, N', and N^*, respectively, the numbers of elements in s, S', and S^*.

The interchange between the stimulus sample and the remainder of the population, that is, between s and S', is assumed to occur at a constant rate over time. Specifically, we assume that during an interval Δt, which is just long enough to permit the interchange of a single element between s and S', there is probability g that such an interchange will occur, the parameter g being constant over time. We shall limit consideration to the special case in which all stimulus elements are equally likely to participate in an interchange. With this restriction, the fluctuation process can be characterized by the difference equation

$$\begin{aligned} f(t+1) &= (1-g)f(t) + g\left\{f(t)\left(1-\frac{1}{N}\right) + [1-f(t)]\frac{1}{N'}\right\} \\ &= \left[1 - g\left(\frac{1}{N} + \frac{1}{N'}\right)\right]f(t) + \frac{g}{N'}, \end{aligned} \tag{67}$$

where $f(t)$ denotes the probability that any given element of S^* is in s at time t. This recursion can be solved by standard methods to yield the explicit formula

$$\begin{aligned} f(t) &= \frac{N}{N^*} - \left[\frac{N}{N^*} - f(0)\right]\left[1 - g\left(\frac{1}{N} + \frac{1}{N'}\right)\right]^t \\ &= J - [J - f(0)]a^t, \end{aligned} \tag{68}$$

where $J = N/N^*$, the proportion of all the elements in the sample, and $a = 1 - g(1/N + 1/N')$.

Equation 68 can now serve as the basis for deriving numerous expressions of experimental interest. Suppose, for example, that at the end of a conditioning (or extinction) period there were j_0 conditioned elements in S and k_0 conditioned elements in S', the momentary probability of a conditioned response thus being $p_0 = j_0/N$. To obtain an expression for probability of a conditioned response after a rest interval of duration t, we proceed as follows. For each conditioned element in S at the beginning of the interval, we need only set $f(0) = 1$ in Eq. 68 to obtain the probability that the element is in S at time t. Similarly, for a conditioned element initially in S' we set $f(0) = 0$ in Eq. 68. Combining the two types, we obtain for the expected number of conditioned elements in S at time t

$$j_0[J - (J-1)a^t] + k_0 J(1 - a^t) = (j_0 + k_0)J - [(j_0 + k_0)J - j_0]a^t.$$

Dividing by N (and noting that $J = N/N^*$) we have, then, for the probability of a conditioned response at time t

$$\begin{aligned} p_t &= \frac{j_0 + k_0}{N^*} - \left[\frac{j_0 + k_0}{N^*} - p_0\right]a^t \\ &= p_0{}^* - (p_0{}^* - p_0)a^t, \end{aligned} \tag{69}$$

where $p_0{}^*$ and p_0 denote the proportion of conditioned elements in the total population S^* and the initial proportion in S, respectively. If the rest interval begins after a conditioning period, we will ordinarily have $p_0 > p_0{}^*$ in which case Eq. 69 describes a decreasing function (forgetting, or spontaneous regression). If the rest interval begins after an extinction period, we will have $p_0 < p_0{}^*$, in which case Eq. 69 describes an increasing function (spontaneous recovery). The manner in which cases of spontaneous regression or recovery depend on the amount and spacing of previous acquisition or extinction has been discussed in detail elsewhere (Estes, 1955a).

APPLICATION TO THE *RTT* EXPERIMENT. We noted in the preceding section that the fixed-sample-size model could not provide a generally satisfactory account of *RTT* experiments because it did not allow for the retention loss usually observed between the first and second tests. It seems reasonable that this defect might be remedied by removing the restriction on independent sampling. To illustrate application of the more general model with provision for stimulus fluctuation, we again consider the case of an *RTT* experiment in which the probability of a correct response is negligible before the reinforced trial (and also on later trials if learning has not occurred). Letting t_1 and t_2 denote the intervals between R and T_1 and between T_1 and T_2, respectively, we may obtain the following basic expressions by setting $f(0)$ equal to 1 or 0, as appropriate, in Eq. 68: For the probability that an element sampled on R is sampled again on T_1,

$$f_1 = J + (1 - J)a^{t_1};$$

for the probability that an element sampled on T_1 is sampled again on T_2,

$$f_2 = J + (1 - J)a^{t_2};$$

and for the probability that an element not sampled on T_1 is sampled on T_2,

$$f_3 = J(1 - a^{t_2}).$$

Assuming now that $N = 1$, so that we are dealing with a generalized form of the pattern model, we can write the probabilities of the four combinations of correct and incorrect responses on T_1 and T_2 in terms of the conditioning parameter c and the parameters f_i:

$$\begin{aligned} p_{00} &= cf_1f_2, \\ p_{01} &= cf_1(1 - f_2), \\ p_{10} &= c(1 - f_1)f_3, \\ p_{11} &= 1 - c + c(1 - f_1)(1 - f_3), \end{aligned} \tag{70}$$

where, as before, the subscripts 0 and 1 denote correct responses and errors, respectively. As they stand, Eqs. 70 are not suitable for application to data because there are too many parameters to be estimated. This difficulty could be surmounted by adding a third test trial, for then the resulting eight observation equations

$$p_{000} = cf_1 f_2^{\,2},$$
$$p_{001} = cf_1 f_2(1 - f_2),$$
$$p_{010} = cf_1(1 - f_2)f_3,$$

etc., would permit overdetermination of the four parameters. In the case of some published studies (e.g., Estes, 1961b) the data can be handled quite well on the assumption that f_1 is approximately unity, in which case Eqs. 70 reduce to

$$p_{00} = cf_2,$$
$$p_{01} = c(1 - f_2),$$
$$p_{10} = 0,$$
$$p_{11} = 1 - c.$$

In the general case of Eqs. 70 some predictions can be made without knowing the exact parameter values. It has been noted in published studies (Estes, Hopkins, & Crothers, 1960; Estes, 1961b) that the observed proportion p_{01} is generally larger than p_{10}. Taking the difference between the theoretical expressions for these quantities, we have

$$\begin{aligned} p_{01} - p_{10} &= cf_1(1 - f_2) - c(1 - f_1)f_3 \\ &= c[J + (1 - J)a^{t_1}](1 - J)(1 - a^{t_2}) \\ &\qquad - c(1 - J)(1 - a^{t_1})J(1 - a^{t_2}) \\ &= c(1 - J)(1 - a^{t_2})[J + (1 - J)a^{t_1} - J(1 - a^{t_1})] \\ &= c(1 - J)(1 - a^{t_2})a^{t_1}, \end{aligned}$$

which obviously must be equal to or greater than zero. The experiments cited above have in all cases had $t_1 < t_2$ and therefore $f_1 > f_2$. Since f_2, which is directly estimated by the proportions of instances in which correct responses on T_1 are repeated on T_2, has ranged from about 0.6 to 0.9 in these experiments (and f_1 must be larger), it is clear that p_{10}, the probability of an incorrect followed by a correct response, should be relatively small. This theoretical prediction accords well with observation.

Numerous predictions can be generated concerning the effects of varying the durations of t_1 and t_2. The probability of repeating a correct response from T_1 to T_2, for example, should depend solely on the parameter f_2, decreasing as t_2 increases (and f_2 therefore decreases). The probability of a correct response on T_2 following an incorrect response on T_1 should depend most strongly on f_3, increasing as t_2 (and therefore f_3) increases.

The over-all proportion correct per test should, of course, decrease from T_1 to T_2 (although the difference between proportions on T_1 and T_2 tends to zero as t_1 becomes large). Data relevant to these and other predictions are available in studies by Estes, Hopkins, and Crothers (1960), Peterson, Saltzman, Hillner, and Land (1962), and Witte (R. Witte, personal communication). The predictions concerning effects of variation of t_2 are well confirmed by these studies. Results bearing on predictions concerning variation in t_1 are not consistent over the set of experiments, possibly because of artifacts arising from item selection (discussed by Peterson et al., 1962).

APPLICATION TO THE SIMPLE NONCONTINGENT CASE. We restrict consideration to the special case of $N = 1$; thus we are dealing with a variant of the pattern model in which the pattern sampled on any trial is the one most likely to be sampled on the next trial. No new concepts are required beyond those introduced in connection with the *RTT* experiment, but it is convenient to denote by a single symbol, say g, the probability that the stimulus pattern sampled on any trial n is exchanged for another pattern on trial $n + 1$. In terms of this notation,

$$g = 1 - f_1 = (1 - J)(1 - a^t) = \left(1 - \frac{1}{N^*}\right)(1 - a^t),$$

where t is now taken to denote the intertrial interval. Also, we denote by $u_{1m,n}$ the probability of the state of the organism in which m stimulus patterns are conditioned to the A_1-response and one of these is sampled and by $u_{0m,n}$ the probability that m patterns are conditioned to A_1 but a pattern conditioned to A_2 is sampled. Obviously

$$p_n = \sum_{m=0}^{N^*} u_{1m,n},$$

where, as usual, p_n denotes the probability of the A_1-response on trial n.

Now we can write expressions for trigram probabilities, following essentially the same reasoning used before in the case of the pattern model with independent sampling. For the joint event $A_1E_1A_1$ we obtain

$$\begin{aligned}\Pr(A_{1,n+1}E_{1,n}A_{1,n}) &= \pi \sum_m u_{1m,n}\left[1 - g + g\frac{m-1}{N'}\right] \\ &= \pi\left[\left(1 - g - \frac{g}{N'}\right)p_n + g\sum_m u_{1m,n}\frac{m}{N'}\right],\end{aligned}$$

for if an element conditioned to A_1 is sampled on trial n then with probability $1 - g$ it is resampled and with probability $g[(m - 1)/N']$

it is replaced by another element conditioned to A_1; in either event an A_1-response must occur on trial $n + 1$. If the abbreviations $U_n = \sum_m u_{1m,n}(m/N')$ and $V_n = \sum_m u_{0m,n}(m/N')$ are used, the trigram probabilities can be written in relatively compact form:

$$\begin{aligned}
\Pr(A_{1,n+1}E_{1,n}A_{1,n}) &= \pi\left[\left(1 - g - \frac{g}{N'}\right)p_n + gU_n\right],\\
\Pr(A_{1,n+1}E_{2,n}A_{1,n}) &= (1 - \pi)\left\{\left[(1 - c)(1 - g) - \frac{g}{N'}\right]p_n + gU_n\right\},\\
\Pr(A_{1,n+1}E_{1,n}A_{2,n}) &= \pi[c(1 - g)(1 - p_n) + gV_n],\\
\Pr(A_{1,n+1}E_{2,n}A_{2,n}) &= (1 - \pi)gV_n, \qquad (71)\\
\Pr(A_{2,n+1}E_{1,n}A_{1,n}) &= \pi g\left[\left(1 + \frac{1}{N'}\right)p_n - U_n\right],\\
\Pr(A_{2,n+1}E_{2,n}A_{1,n}) &= (1 - \pi)\left[\left(c - cg + g + \frac{g}{N'}\right)p_n - gU_n\right],\\
\Pr(A_{2,n+1}E_{1,n}A_{2,n}) &= \pi[(1 - c + cg)(1 - p_n) - gV_n],\\
\Pr(A_{2,n+1}E_{2,n}A_{2,n}) &= (1 - \pi)[1 - p_n - gV_n].
\end{aligned}$$

The chief difference between these expressions and the corresponding ones for the independent sampling models is that sequential effects now depend on the intertrial interval. Consider, for example, the first two of Eqs. 71, involving repetitions of response A_1. It will be noted that both expressions represent linear combinations of p_n and U_n, with the relative contribution of p_n increasing as the intertrial interval (and therefore g) decreases. Also, it is apparent from the defining equations for p_n and U_n that $p_n \geqslant U_n$, with equality obtaining only in the special cases in which both are equal to unity or both equal to zero. Therefore, the probability of a repetition is inversely related to the intertrial interval. In particular, the probability that a correct A_1- or A_2-response will be repeated tends to unity in the limit as the intertrial interval goes to zero. When the intertrial interval becomes large, the parameter g approaches $1 - 1/N^*$ and Eqs. 71 reduce to those of a pattern model with N elements and independent sampling.

Summing the first four of Eqs. 71, we obtain a recursion for probability of the A_1-response:

$$p_{n+1} = \left(1 - c - g - \frac{g}{N'} + cg\right)p_n + c(1 - g)\pi + g(U_n + V_n).$$

Now, although a full proof would be quite involved, it is not hard to show heuristically that the asymptote is independent of the intertrial interval. We note first that asymptotically we have

$$\begin{aligned} U_n &= \sum_m u_{1m} \frac{m}{N'} \\ &= \sum_m u_m \frac{m}{N^*} \frac{m}{N'} \\ &= \frac{N^*}{N'} \sum_m \left(\frac{m}{N^*}\right)^2 u_m \\ &= \frac{N^*}{N'} \alpha_{2,n}, \end{aligned}$$

where u_m is the probability that m elements are conditioned to A_1. The substitution of $u_m(m/N^*)$ for u_{1m} is possible in view of the intuitively evident fact that, asymptotically, the probability that an element conditioned to A_1 will constitute the trial sample is simply equal to the proportion of such elements in the total population. Substituting into the recursion for p_n in terms of this relation, and the analogous one for V_n,

$$V_n = \frac{N^*}{N'} (p_n - \alpha_{2,n}),$$

we obtain

$$\begin{aligned} p_{n+1} &= \left(1 - c - g - \frac{g}{N'} + cg\right) p_n + c(1 - g)\pi + g \frac{N^*}{N'} p_n \\ &= (1 - c + cg)p_n + c(1 - g)\pi, \end{aligned}$$

the simplification in the last line having been effected by means of the identity

$$-g - \frac{g}{N'} = -g\left(\frac{N' + 1}{N'}\right) = -g \frac{N^*}{N'}.$$

Setting $p_{n+1} = p_n = p_\infty$ and solving for p_∞, we arrive at the tidy outcome

$$p_\infty = (1 - c + cg)p_\infty + c(1 - g)\pi,$$

whence

$$p_\infty = \pi.$$

The recursion in p_n can be solved, but the resulting formula expressing p_n as a function of n and the parameters is too cumbersome to yield much useful information by visual inspection. It seems intuitively obvious that for $g < 1 - 1/N^*$ (i.e., for any but very long intertrial intervals) the learning curve will rise more sharply on early trials than the corresponding curve for the independent sampling case. This is so because only sampled elements can undergo conditioning, and, once sampled, an element is more likely to be resampled the shorter the intertrial interval. However, the curves for longer and shorter intervals must cross ultimately, with the curve for the longer interval approaching asymptote more rapidly on later trials (Estes, 1955b). If $\pi = 1$, the total number of errors expected during learning must be independent of the intertrial interval because each initially unconditioned element will continue to produce an error each time it is sampled until it is finally conditioned, and the probability of any specified number of errors before conditioning depends only on the value of the conditioning parameter c. Similarly, if π is set equal to 0 after a conditioning session, the total number of conditioned responses during extinction is independent of the intertrial interval.

4.3 The Linear Model as a Limiting Case

For those experiments in which the available stimuli are the same on all trials the possibility arises of using a model that suppresses the concept of stimuli. In such a "pure" reinforcement model the learning assumptions specify directly how response probability changes on a reinforced trial. By all odds the most popular models of this sort are those which assume probability of a response on a given trial to be a linear function of the probability of that response on the previous trial.[12]

The so-called "linear models" received their first systematic treatment by Bush and Mosteller (1951a, 1955) and have been investigated and developed further by many others. We shall be concerned only with a certain class of linear models based on a single learning parameter θ. A more extensive analysis of this class of linear models has been given in Estes & Suppes (1959a).

The linear theory is formulated for the probability of a response on trial $n + 1$, *given* the entire preceding sequence of responses and reinforcements.[13] Let x_n be the sequence of responses and reinforcements of a given subject through trial n; that is, x_n is a sequence of length $2n$ with entries in the odd positions indicating responses and entries in the even positions indicating reinforcements. The axioms of the linear model are as follows.

Linear Axioms

For every i, i' and k such that $1 \leqslant i, i' \leqslant r$ and $0 \leqslant k \leqslant r$:

L1. *If* $\Pr(E_{i,n}A_{i',n}x_{n-1}) > 0$, *then*

$$\Pr(A_{i,n+1} \mid E_{i,n}A_{i',n}x_{n-1}) = (1 - \theta)\Pr(A_{i,n} \mid x_{n-1}) + \theta.$$

[12] For a discussion of this general class of "incremental" models see Chapter 9 by Sternberg in this volume.

[13] In the language of stochastic processes we have a chain of infinite order.

L2. *If* $\Pr(E_{k,n}A_{i',n}x_{n-1}) > 0$, $k \neq i$ *and* $k \neq 0$, *then*
$$\Pr(A_{i,n+1} \mid E_{k,n}A_{i',n}x_{n-1}) = (1-\theta)\Pr(A_{i,n} \mid x_{n-1}).$$

L3. *If* $\Pr(E_{0,n}A_{i',n}x_{n-1}) > 0$, *then*
$$\Pr(A_{i,n+1} \mid E_{0,n}A_{i',n}x_{n-1}) = \Pr(A_{i,n} \mid x_{n-1}).$$

By Axiom L1, if the reinforcing event E_i, corresponding to response A_i, occurs on trial n, then (regardless of the response occurring on trial n) the probability of A_i increases by a linear transform of the old value. By L2, if some reinforcing event other than E_i occurs on trial n, then the probability of A_i decreases by a linear transform of its old value; and by L3 occurrence of the "neutral" event E_0 leaves response probabilities unchanged. The axioms may be written more compactly in terms of the probability $p_{xi,n}$ that a subject identified with sequence x makes an A_i response on trial n:

1. If the subject receives an E_i-event on trial n,
$$p_{xi,n+1} = (1-\theta)p_{xi,n} + \theta.$$
2. If the subject receives an E_k-event ($k \neq i$ and $k \neq 0$) on trial n,
$$p_{xi,n+1} = (1-\theta)p_{xi,n}.$$
3. If the subject receives an E_0-event on trial n,
$$p_{xi,n+1} = p_{xi,n}.$$

From a mathematical standpoint it is important to note that for the linear model the response probability associated with a particular subject is free to vary continuously over the entire interval from 0 to 1, since this probability undergoes linear transformations as a result of reinforcement. Consequently, if we wish to interpret changes in response probability as transitions among states of a Markov process, we must deal with a continuous-state space. Thus the Markov interpretation is of little practical value for calculational purposes. In stimulus sampling models response probability is defined in terms of the proportion of stimuli conditioned; since the set of stimuli is finite, so also is the set of values taken on by the response probability of any individual subject. It is this finite character of stimulus sampling models that makes possible the extremely useful interpretation of the models as finite Markov chains.

An inspection of the three axioms for the linear model indicates that they have the same general form as Eqs. 65, which describe changes in response probability for the fixed-sample-size component model; that is, if we let $\theta = cs/N$, then the two sets of rules are similar. As might be expected from this observation, many of the predictions generated by the two models are identical when $\theta = cs/N$. For example, in the simple non-contingent situation the mean learning curve for the linear model is

$$\Pr(A_{1,n}) = \pi - [\pi - \Pr(A_{1,1})](1 - \theta)^{n-1}, \tag{72}$$

which is the same as that of the component model (see Estes & Suppes, 1959a, for a derivation of results for the linear model). However, the two models are not identical in all respects, as is indicated by a comparison of the asymptotic variances of the response distributions. For the linear model

$$\sigma_\infty{}^2 = \pi(1 - \pi)\frac{\theta}{2 - \theta},$$

as contrasted to Eq. 63 for the component model. However, as already noted in connection with Eq. 63, in the limit (as $N \to \infty$) the $\sigma_\infty{}^2$ for the component model equals the predicted value for the linear model.

The last result suggests that the component model may converge to the linear process as $N \to \infty$. This conjecture is substantially correct; it can be shown that in the limit both the fixed-sample-size model and the independent sampling model approach the linear model for an extremely broad class of assumptions governing the sampling of elements. The derivation of the linear model from component models holds for any reinforcement schedule, for any finite number r of responses, and for every trial n, not simply at asymptote. The proof of this convergence theorem is lengthy and it is not presented here. However, the proof depends on the fact that the variance of the sampling distribution for any statistic of the trial sample approaches 0 as N becomes large. A proof of the convergence theorem is given by Estes and Suppes (1959b). Kemeny and Snell (1957) also have considered the problem but their proof is restricted to the two-choice noncontingent situation at asymptote.

COMPARISON OF THE LINEAR AND PATTERN MODELS. The same limiting result does not, of course, hold for the pattern model discussed in Sec. 2. For the pattern model only one element is sampled on each trial, and it is obvious that as $N \to \infty$ the learning effect of this sampling scheme would diminish to zero. For experimental situations in which both the linear model and the pattern model appear to be applicable it is important to derive differential predictions from the two models that, on empirical grounds, will permit the researcher to choose between them. To this end we display a few predictions for the linear model applied to both the *RTT* situation and the simple two-response noncontingent situation; these results will be compared with the corresponding equations for the pattern model.

For simplicity let us assume that in the case of the *RTT* situation the likelihood of a correct response by guessing is negligible on all trials. Then, according to the linear model, the probability of a reinforced response changes in accordance with the equation

$$p_{n+1} = (1 - \theta)p_n + \theta.$$

In the present application the probability of a correct response on the first trial (the R trial) is zero, hence the probability of a correct response on the first test trial is simply θ. No reinforcement is given on T_1, and consequently the probability of a correct response does not change between T_1 and T_2. Therefore, p_{00}, the probability of a correct response on both T_1 and T_2 (as defined in connection with Eq. 55) is θ^2. Similarly, we obtain $p_{01} = p_{10} = \theta(1 - \theta)$ and $p_{11} = (1 - \theta)^2$. Some relevant data are presented in Table 6 (from Estes, 1961b). They represent joint response

Table 6 Observed Joint Response Proportions for *RTT* Experiment and Predictions from Linear Retention-Loss Model and Sampling Model

	Observed Proportion	Retention-Loss Model	Sampling Model
p_{00}	0.238	0.238	0.238
p_{01}	0.147	0.238	0.152
p_{10}	0.017	0.018	0
p_{11}	0.598	0.506	0.610

proportions for 40 subjects, each tested on 15 paired associate items of the type described in Sec. 1.1, the *RTT* design applied to each item. In order to minimize the probability of correct responses occurring by guessing, these items were introduced (one per trial) into a larger list, the composition of which changed from trial to trial. A critical item introduced on trial n received one reinforcement (paired presentation of stimulus and response members), followed by a test (presentation of stimulus alone) on trial n and trial $n + 1$, after which it was dropped from the list.

From an inspection of the data column of Table 6 it is obvious that the simple linear model cannot handle these proportions. It suffices to note that the model requires $p_{01} = p_{10}$, whereas the difference between these two entries in the data column is quite large.

One might try to preserve the linear model by arguing that the pattern of observed results in Table 6 could have arisen as an artifact. If, for example, there are differences in difficulty among items (or, equivalently, differences in learning rate among subjects), then the instances of incorrect response on T_1 would predominately represent smaller θ-values than instances of correct responses. On this account it might be expected that the predicted proportion of correct following incorrect responses would be smaller than that allowed for under the "equal θ" assumption and therefore that the linear model might not actually be incompatible with the data of Table 6. We can easily check the validity of such an argument. Suppose that parameter θ_i is associated with a proportion f_i of the items (or subjects). Then in each case in which θ_i is applicable the probability of a correct response on T_1 followed by an error on T_2 is $\theta_i(1 - \theta_i)$. Clearly, then, p_{01} estimated from a group of items described by differences in θ would be

$$p_{01} = \sum_i f_i\theta_i(1 - \theta_i).$$

But a similar argument yields

$$p_{10} = \sum_i f_i(1 - \theta_i)\theta_i.$$

Since, again, the expressions for p_{10} and p_{01} are equal for all distributions of θ_i, it is clear that individual differences in learning rates alone could not account for the observed results.

A related hypothesis that might seem to merit consideration is that of individual differences in rates of forgetting. Since the proportion of correct responses on T_2 is less than that on T_1, there is evidently some retention loss, and differences among subjects, or items, in susceptibility to this retention loss might be a source of bias in the data. The hypothesis can be formulated in the linear model as follows: the probability of the correct response on T_1 is equal to θ; if, however, there is a retention loss, then the probability of a correct response on T_2 will have declined to some value ρ, such that $\rho < \theta$. If there are individual differences in amount of retention loss, then we should again categorize the population of subjects and items into subgroups, with a proportion f_i of the subjects characterized by retention parameter ρ_i. Theoretical expressions for p_{jk} can be derived for such a population by the same method used in the preceding case; the results are

$$p_{00} = \theta \sum_i f_i\rho_i,$$

$$p_{01} = \theta \sum_i f_i(1 - \rho_i),$$

$$p_{10} = (1 - \theta) \sum_i f_i\rho_i,$$

$$p_{11} = (1 - \theta) \sum_i f_i(1 - \rho_i).$$

This time the expressions for p_{10} and p_{01} are different; with a suitable choice of parameter values, they could accommodate the difference between the observed proportions p_{01} and p_{10}. However, another difficulty remains. To obtain a near-zero value for p_{10} would require either a θ near unity, which would be incompatible with the observed proportion of 0.385 correct on T_1, or a value of $\sum_i f_i\rho_i$ near zero, which would be incompatible with the observed proportion of 0.255 correct on T_2. Thus we have no support for the hypothesis that individual differences in amount of retention loss might account for the pattern of empirical values.

We could go on in a similar fashion and examine the results of supplementing the original linear model by hypotheses involving more complex combinations or interactions of possible sources of bias (see Estes, 1961b). For example, we might assume that there are large individual differences in both learning and retention parameters. But, even with this latitude, it would not be easy to adjust the linear model to the *RTT* data. Suppose that we admit different learning parameters, θ_1 and θ_2, and different retention parameters, ρ_1 and ρ_2, the combination $\theta_1\rho_1$ obtaining for half the items and the combination $\theta_2\rho_2$ for the other half. Now the p_{ij} formulas become

$$p_{00} = \frac{\theta_1\rho_1 + \theta_2\rho_2}{2},$$

$$p_{01} = \frac{\theta_1(1 - \rho_1) + \theta_2(1 - \rho_2)}{2},$$

$$p_{10} = \frac{(1 - \theta_1)\rho_1 + (1 - \theta_2)\rho_2}{2},$$

$$p_{11} = \frac{(1 - \theta_1)(1 - \rho_1) + (1 - \theta_2)(1 - \rho_2)}{2}.$$

From the data column of Table 6 the proportions of correct responses on the first and second test trials are $p_{0-} = 0.385$ and $p_{-0} = 0.255$, respectively. Adding the first and second of the foregoing equations to obtain the theoretical expression for p_{0-} and the first and third equations to get p_{-0}, we have

$$p_{0-} = \frac{\theta_1 + \theta_2}{2}$$

and

$$p_{-0} = \frac{\rho_1 + \rho_2}{2}.$$

Equating theoretical and observed values, we obtain the constraints

$$\theta_1 + \theta_2 = 0.770$$
$$\rho_1 + \rho_2 = 0.510,$$

which should be satisfied by the parameter values. If the proportion p_{00} in Table 6 is to be predicted correctly, we must have

$$\frac{\theta_1\rho_1 + \theta_2\rho_2}{2} = 0.238,$$

or, substituting from the two preceding equations,

$$\theta_1\rho_1 + (0.77 - \theta_1)(0.51 - \rho_1) = 0.476,$$

which may be solved for θ_1:

$$\theta_1 = \frac{0.083 + 0.77\rho_1}{2\rho_1 - 0.51}.$$

Now the admissible range of parameter values can be further reduced. For the right-hand side of this last equation to have a value between 0 and 1, ρ_1 must be greater than 0.48; so we have the relatively narrow bounds on the parameters ρ_i

$$0.48 \leqslant \rho_1 \leqslant 0.51$$
$$0 \leqslant \rho_2 \leqslant 0.03.$$

Using these bounds on ρ_1, we find from the equation expressing θ_1 as a function of ρ_1 that θ_1 must in turn satisfy $0.93 \leqslant \theta_1 \leqslant 1.0$. But now the model is in trouble, for, in order to satisfy the constraint $\theta_1 + \theta_2 = 0.77$, θ_2 would have to be negative (and the correct response probabilities for half of the items on T_1 would also be negative). About the best we can do, without allowing "negative probabilities," is to use the limits we have obtained for ρ_1, ρ_2, and θ_1 and arbitrarily assign a zero or small positive value to θ_2. Choosing the combination $\theta_1 = 0.95$, $\theta_2 = 0.01$, $\rho_1 = 0.5$, and $\rho_2 = 0.01$, we obtain the theoretical values listed for the linear model in Table 6. By introducing additional assumptions or additional parameters, we could improve the fit of the linear model to these data, but there would seem to be little point in doing so. The refractoriness of the data to description by any reasonably simple form of the model suggests that perhaps the learning process is simply not well represented by the type of growth function embodied in the linear model.

By contrast, these data can be quite readily handled by the stimulus fluctuation model developed in the preceding section. Letting $f_1 = 1$ in Eqs. 70 and using the estimates $c = 0.39$ and $f_2 = 0.61$, we obtain the theoretical values listed under "Sampling Model" in Table 6. We would not, of course, claim that the sampling model had been rigorously tested, since two parameters had to be estimated and there are only three degrees of freedom in this set of data. However, the model does seem more promising than any of the variants of the linear model that have been investigated. More stringent tests of the sampling model can readily be obtained by running similar experiments with longer sequences of test trials, since predictions concerning joint response proportions over blocks of three or more test trials can be generated without additional assumptions.

ADDITIONAL COMPARISONS BETWEEN THE LINEAR AND PATTERN MODEL. We now turn to a few comparisons between the linear model and the multi-element pattern model for the simple noncontingent situation. First of all, we note that the mean learning curves for the two models (as given in Eq. 37 and Eq. 72) are identical if we let $c/N = \theta$. However, the expressions for the variance of the asymptotic response distribution are different; for the linear model $\sigma_\infty{}^2 = \pi(1 - \pi)[\theta/(2 - \theta)]$, whereas for the pattern model $\sigma_\infty{}^2 = \pi(1 - \pi)(1/N)$. This difference is reflected in another prediction that provides a more direct experimental test of the two models. It concerns the asymptotic variance of the distribution of the number of A_1-responses in a block of K trials which we denote Var $(\bar{\mathbf{A}}_K)$. For the linear model (cf. Estes & Suppes, 1959a),

$$\mathrm{Var}\,(\bar{\mathbf{A}}_K) = \pi(1 - \pi)\left\{\frac{K(4 - 3\theta)}{2 - \theta} - \frac{2(1 - \theta)}{(2 - \theta)\theta}[1 - (1 - \theta)^K]\right\}.$$

For the pattern model, by Eq. 42,

$$\mathrm{Var}\,(\bar{\mathbf{A}}_K) = \pi(1 - \pi)\left\{K + \frac{2K(1 - c)}{c} - \frac{2(1 - c)N}{c^2}\left[1 - \left(1 - \frac{c}{N}\right)^K\right]\right\}.$$

Note that, for $c = \theta$, the variance for the pattern model is larger than for the linear model. However, for the case of $\theta = c/N$ the variance for the pattern model can be larger or smaller than for the linear model depending on the particular values of c and N.

Finally, we present certain asymptotic sequential predictions for the linear model in the noncontingent situation; namely

$$\begin{aligned}
\lim \Pr\,(A_{1,n+1} \mid E_{1,n}A_{1,n}) &= (1 - \theta)a + \theta \\
\lim \Pr\,(A_{1,n+1} \mid E_{2,n}A_{1,n}) &= (1 - \theta)a \\
\lim \Pr\,(A_{1,n+1} \mid E_{1,n}A_{2,n}) &= (1 - \theta)b + \theta \\
\lim \Pr\,(A_{1,n+1} \mid E_{2,n}A_{2,n}) &= (1 - \theta)b
\end{aligned}$$

where

$$a = \pi + \frac{\theta(1 - \pi)}{2 - \theta} \quad \text{and} \quad b = \pi - \frac{\theta\pi}{2 - \theta}.$$

These predictions are to be compared with Eq. 34 for the pattern model. In the case of the pattern model we note that $\Pr\,(A_1 \mid E_1A_1)$ and $\Pr\,(A_1 \mid E_2A_2)$ depend only on π and N, whereas $\Pr\,(A_1 \mid E_2A_1)$ and $\Pr\,(A_1 \mid E_1A_2)$ depend on π, N, and c. In contrast, all four sequential probabilities depend on π and θ in the linear model. For comparisons between the linear model and the pattern model in application to two-choice data, the reader is referred to Suppes & Atkinson (1960).

4.4 Applications to Multiperson Interactions

In this section we apply the linear model to experimental situations involving multiperson interactions in which the reinforcement for any given subject depends both on his response and on the responses of other subjects. Several recent investigations have provided evidence indicating the fruitfulness of this line of development. For example, Bush and Mosteller (1955) have analyzed a study of imitative behavior in terms of their linear model, and Estes (1957a), Burke (1959, 1960), and Atkinson and Suppes (1958) have derived and tested predictions from linear models for behavior in two- and three-person games. Suppes and Atkinson (1960) have also provided a comparison between pattern models and linear models for multiperson experiments and have extended the analysis to situations involving communication between subjects, monetary payoff, social pressure, economic oligopolies, and related variables.

The simple two-person game has particular advantages for expository purposes, and we use this situation to illustrate the technique of extending the linear model to multiperson interactions. We consider a situation which, from the standpoint of game theory (see, e.g., Luce & Raiffa, 1957), may be characterized as a game in normal form with a finite number of strategies available to each player. Each play of the game constitutes a trial, and a player's choice of a strategy for a given trial corresponds to the selection of a response. To avoid problems having to do with the measurement of utility (or from the viewpoint of learning theory, problems of reward magnitude), we assume a unit reward that is assigned on an all-or-none basis. Rules of the game require the two players to exhibit their choices simultaneously on all trials (as in a game of matching pennies), and each player is informed that, given the choice of the other player on the trial, there is exactly one choice leading to the unit reward.

We designate the two players as A and B and let A_i $(i = 1, \ldots, r)$ and B_j $(j = 1, \ldots, r')$ denote the responses available to the two players. The set of reinforcement probabilities prescribed by the experimenter may be represented in a matrix (a_{ij}, b_{ij}) analogous to the "payoff matrix" familiar in game theory. The number a_{ij} represents the probability of Player A being correct on any trial of the experiment, given the response pair A_iB_j; similarly, b_{ij} is the probability of Player B being correct, given the response pair A_iB_j. For example, consider the matrix

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} \frac{1}{2}, \frac{1}{2} & 1, 0 \\ 1, 0 & 0, 1 \end{bmatrix}. \end{array}$$

When both subjects make Response 1, each has probability $\frac{1}{2}$ of receiving reward; when both make Response 2, then only Player B receives reward; when either of the other possible response pairs occurs (i.e., A_2B_1 or A_1B_2), then only Player A receives reward. It should be emphasized that, although one usually thinks of one player winning and the other losing on any given play of a game, this is not a necessary restriction on the model. In theory, and in experimental tests of the theory, it is quite possible to permit both or neither of the players to be rewarded on any trial. However, to provide a relatively simple theoretical interpretation of reinforcing events, it is essential that on a nonrewarded trial the player be informed (or led to infer) that some other choice, had he made it under the same circumstances, would have been successful. We return to this point later.

Let $E_i^{(A)}$ denote the event of reinforcing the A_i response for Player A and $E_j^{(B)}$ the event of reinforcing the B_j response for Player B. To simplify our analysis, we consider the case in which each subject has only two response alternatives, and we define the probability of occurrence of a particular reinforcing event in terms of the payoff parameters as follows (for $i \neq i'$ and $j \neq j'$):

$$\begin{aligned} a_{ij} &= \Pr(E_i^{(A)} \mid A_{i,n}B_{j,n}) & b_{ij} &= \Pr(E_j^{(B)} \mid A_{i,n}B_{j,n}) \\ 1 - a_{ij} &= \Pr(E_{i'}^{(A)} \mid A_{i,n}B_{j,n}) & 1 - b_{ij} &= \Pr(E_{j'}^{(B)} \mid A_{i,n}B_{j,n}). \end{aligned} \tag{73}$$

For example, if Player A makes an A_1-response and is rewarded, then an $E_1^{(A)}$ occurs; however, if an A_1 is made and no reward occurs, then we assume that the other response is reinforced, that is, an $E_2^{(A)}$ occurs.

Finally, one last definition to simplify notation. We denote Player A's response probability by α and Player B's by β, and we denote by γ the joint probability of an A_1- and B_1-response. Specifically,

$$\alpha_n = \Pr(A_{1,n}), \qquad \beta_n = \Pr(B_{1,n}), \qquad \gamma_n = \Pr(A_{1,n}B_{1,n}). \tag{74}$$

We now derive a theorem that provides recursive expressions for α_n and β_n and points up a property of the model that greatly complicates the mathematics, namely, that both α_{n+1} and β_{n+1} depend on the joint probability $\gamma_n = \Pr(A_{1,n}B_{1,n})$. The statement of the theorem is as follows:

$$\begin{aligned} \alpha_{n+1} = {} & [1 - \theta_A(2 - a_{12} - a_{22})]\alpha_n + \theta_A(a_{22} - a_{21})\beta_n \\ & + \theta_A(a_{11} + a_{21} - a_{12} - a_{22})\gamma_n + \theta_A(1 - a_{22}) \end{aligned} \tag{75a}$$

$$\begin{aligned} \beta_{n+1} = {} & [1 - \theta_B(2 - b_{21} - b_{22})]\beta_n + \theta_B(b_{22} - b_{12})\alpha_n \\ & + \theta_B(b_{11} + b_{12} - b_{21} - b_{22})\gamma_n + \theta_B(1 - b_{22}), \end{aligned} \tag{75b}$$

where θ_A and θ_B are the learning parameters for players A and B. In the proof of this theorem it will suffice to derive the difference equation for α_{n+1}, since the derivation for β_{n+1} is identical. To begin with, from

Axioms L1 and L2 we can easily show that the general form of a recursion for α_n is

$$\alpha_{n+1} = (1 - \theta_A)\alpha_n + \theta_A \Pr(E_{1,n}^{(A)}).$$

The term $\Pr(E_{1,n}^{(A)})$ can then be expanded to

$$\begin{aligned}\Pr(E_{1,n}^{(A)}) &= \sum_{i,j} \Pr(E_{1,n}^{(A)} A_{i,n} B_{j,n}) \\ &= \sum_{i,j} \Pr(E_{1,n}^{(A)} \mid A_{i,n} B_{j,n}) \Pr(A_{i,n} B_{j\ n})\end{aligned}$$

and by Eqs. 73

$$\begin{aligned}\Pr(E_{1,n}^{(A)}) &= a_{11} \Pr(A_{1,n}B_{1,n}) + a_{12} \Pr(A_{1,n}B_{2,n}) \\ &\quad + (1 - a_{21}) \Pr(A_{2,n}B_{1,n}) + (1 - a_{22}) \Pr(A_{2,n}B_{2,n}). \qquad (76)\end{aligned}$$

Next we observe that

$$\begin{aligned}\Pr(A_{1,n}B_{2,n}) &= \Pr(B_{2,n} \mid A_{1,n}) \Pr(A_{1,n}) \\ &= [1 - \Pr(B_{1,n} \mid A_{1,n})] \Pr(A_{1,n}) \qquad (77a) \\ &= \Pr(A_{1,n}) - \Pr(A_{1,n}B_{1,n}).\end{aligned}$$

Similarly,

$$\Pr(A_{2,n}B_{1,n}) = \Pr(B_{1,n}) - \Pr(A_{1,n}B_{1,n}), \qquad (77b)$$

and

$$\begin{aligned}\Pr(A_{2,n}B_{2,n}) &= \Pr(A_{2,n} \mid B_{2,n}) \Pr(B_{2,n}) \\ &= [1 - \Pr(A_{1,n} \mid B_{2,n})] \Pr(B_{2,n}) \\ &= \Pr(B_{2,n}) - \Pr(A_{1,n}B_{2,n}) \qquad (77c) \\ &= 1 - \Pr(B_{1,n}) - \Pr(A_{1,n}) + \Pr(A_{1,n}B_{1,n}).\end{aligned}$$

Substituting into Eq. 76 from Eqs. 77*a*, 77*b*, and 77*c* and simplifying by means of the definitions of α, β, and γ, we obtain

$$\begin{aligned}\Pr(E_{1,n}^{(A)}) &= a_{11}\gamma_n + a_{12}(\alpha_n - \gamma_n) + (1 - a_{21})(\beta_n - \gamma_n) \\ &\quad + (1 - a_{22})(1 - \alpha_n - \beta_n + \gamma_n) \\ &= -(1 - a_{12} - a_{22})\alpha_n + (a_{22} - a_{21})\beta_n \\ &\quad + (a_{11} + a_{21} - a_{12} - a_{22})\gamma_n + (1 - a_{22}).\end{aligned}$$

Substitution of this expression into the general recursion for α_n yields the desired result, which completes the proof.

It has been shown by Lamperti and Suppes (1959) that the limits α, β, and γ exist, whence (letting $\alpha_{n+1} = \alpha_n = \alpha$, $\beta_{n+1} = \beta_n = \beta$ and $\gamma_n = \gamma$ in Eqs. 75*a* and 75*b*) we have two linear relations that are independent of θ_A and θ_B, namely,

$$a\alpha = b\beta + c\gamma + d, \qquad e\beta = f\alpha + g\gamma + h, \qquad (78)$$

where

$$\begin{aligned} a &= 2 - a_{12} - a_{22} & b &= a_{22} - a_{21} \\ c &= a_{11} + a_{21} - a_{12} - a_{22} & d &= 1 - a_{22} \\ e &= 2 - b_{21} - b_{22} & f &= b_{22} - b_{12} \\ g &= b_{11} + b_{12} - b_{21} - b_{22} & h &= 1 - b_{22}. \end{aligned} \tag{79}$$

By eliminating γ from Eqs. 78 we obtain the following linear relation in α and β:

$$(-ag - ce)\alpha + (bg + cf)\beta = ch - dg. \tag{80}$$

Unfortunately, this relationship is one of the few quantitative results that can be directly computed for the linear model. It has, however, the advantageous feature that it is independent of the learning parameters θ_A and θ_B and therefore may be compared directly with experimental data. Application of this result can be illustrated in terms of the game cited earlier in which the payoff matrix takes the form

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ \\ A_2 \end{array} & \begin{bmatrix} \frac{1}{2}, \frac{1}{2} & 1, 0 \\ & \\ 1, 0 & 0, 1 \end{bmatrix}. \end{array}$$

From Eqs. 79 we obtain

$$\begin{array}{llll} a = 1 & b = -1 & c = \frac{1}{2} & d = 1 \\ e = 1 & f = 1 & g = -\frac{1}{2} & h = 0 \end{array}$$

and Eq. 80 becomes

$$(\tfrac{1}{2} - \tfrac{1}{2})\alpha + (\tfrac{1}{2} + \tfrac{1}{2})\beta = \tfrac{1}{2}$$

or $\beta = \frac{1}{2}$. From this result we predict immediately that the long-run proportion of B_1-responses will tend to $\frac{1}{2}$. To derive a prediction for Player A, we substitute the known values of the parameters into the first part of Eq. 78 to obtain

$$\begin{aligned} \alpha &= -\beta + \tfrac{1}{2}\gamma + 1 \\ &= \tfrac{1}{2} + \tfrac{1}{2}\gamma. \end{aligned}$$

Unfortunately we cannot compute γ, the asymptotic probability of the A_1B_1-response pair. However, we know γ is positive, and, since only one half of Player B's responses are B_1's, γ cannot be greater than $\frac{1}{2}$. Therefore we have $0 \leqslant \gamma \leqslant \frac{1}{2}$ and as a result can set definite bounds on the long-run probability of an A_1-response, namely,

$$\tfrac{1}{2} \leqslant \alpha \leqslant \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{3}{4}.$$

Thus we have the basis for a rather exacting experimental test, since the asymptotic predictions for both subjects are parameter-free; that is, they do not depend on the θ-values of either subject or on the initial response probabilities.

Of course, by imposing restrictions on the experimentally determined parameters a_{ij} and b_{ij} a variety of results can be obtained. We limit ourselves to the consideration of one such case: choice of the parameters so that the coefficients of γ_n will vanish in the recursive equations (75*a*) and (75*b*). Specifically, if we let $c = g = 0$ and $af - be \neq 0$, then

$$\begin{aligned} \alpha_{n+1} &= a\alpha_n + b\beta_n + d \\ \beta_{n+1} &= e\beta_n + f\alpha_n + h. \end{aligned} \tag{81}$$

Solutions for this system are well known and can be obtained by a number of different techniques; for a detailed discussion of the problem of obtaining explicit expressions of α_n and β_n for arbitrary n the reader is referred to an article by Burke (1960). We do know, however, that the limits for α_n and β_n exist and are independent of both the initial conditions and θ_A and θ_B. By substituting $\alpha = \alpha_{n+1} = \alpha_n$ and $\beta = \beta_{n+1} = \beta_n$ into the two recursions we obtain

$$\alpha = \frac{bh + df}{af - be}$$

and

$$\beta = \frac{ah + de}{af - be}.$$

The fact that α and β are independent of θ_A and θ_B under the restrictions imposed on the parameters in no way implies that γ is also independent of these quantities.

Equations 81 provide a precise test of the model, and the necessary conditions for this test involve only experimentally manipulable parameters. A great deal of experimental work has been conducted on this restricted problem, and, in general, the correspondence between predicted and observed values has been good; for accounts of this work see Atkinson & Suppes (1958), Burke (1959, 1960), and Suppes & Atkinson (1960).

In conclusion we should mention that all of the predictions presented in this section are identical to those that can be derived from the pattern model of Sec. 2. However, in general, only the grosser predictions, such as those for α_n and β_n, are the same for the two models.

5. DISCRIMINATION LEARNING[14]

The distinction between simple learning and discrimination learning is somewhat arbitrary. By discrimination we refer, roughly speaking, to the

[14] Using the terminology proposed by Bush, Galanter, and Luce in Chapter 2, the class of problems considered in this section would be called "identification-learning" experiments.

process whereby the subject learns to make one response to one of a pair of stimuli and a different response to the other. But there is an element of discrimination in any learning situation. Even in the simplest conditioning experiment the subject learns to make a conditioned response only when the conditioned stimulus is presented, and therefore to do something else when that stimulus is absent. In the paired-associate situation (referred to several times in preceding sections) the subject learns to associate the appropriate member of a response set with each member of a set of stimuli and therefore to "discriminate" the stimuli. The principal basis for differentiation between the two categories of learning seems to be that in the case of discrimination learning the similarity, or communality, between stimuli is a major independent variable; in the case of simple learning stimulus similarity is an extraneous factor to be minimized experimentally and neglected in theory as far as possible.

One of the general strategic assumptions of the type of stimulus-response theory, which has been associated with the development of stimulus sampling models, is that discrimination learning involves a combination of processes, each of which can be studied independently in simpler situations —the learning aspect in experiments on acquisition or extinction and the stimulus relationships in experiments on stimulus generalization or transfer of training. Thus there will be nothing new at the conceptual level in our treatment of discrimination. There is adequate scope for analysis of different types of discriminative situations, but, since our main concern in this section is with methods rather than content, we shall not go far in this direction. We propose only to show how the processes of association and generalization treated in preceding sections enter into discrimination learning, and this can be accomplished by formulating assumptions and deriving results of general interest for a few important cases.

5.1 The Pattern Model for Discrimination Learning

As in the cases of simple acquisition and probability learning, it is sometimes useful in the treatment of discriminative situations to ignore generalization effects among the stimuli involved in an experiment and to regard each stimulus display as a unique pattern. Thus behavior elicited by the stimulus display will depend only on the subject's reinforcement history with respect to that particular pattern. Two important variants of the model arise, depending on whether experimental arrangements do or do not ensure that the subject will sample the entire stimulus display presented on each trial.

Case 1. All cues presented are sampled on each trial. For a classical two-stimulus, two-response discrimination problem (e.g., a Lashley situation in which the rat is differentially rewarded for jumping to a black card and avoiding a grey card) our conceptualization requires a distinction among three types of cues: we denote by S_1 the set of component cues present only in the stimulus situation associated with reinforcement of response A_1, by S_2 the set of cues present only in the situation associated with reinforcement of response A_2, and by S_c the set of cues present in both situations. In the example of the Lashley situation A_1 might be the response of jumping to the left-hand window; A_2, the response of jumping to the right-hand window; S_1, the stimulation present only on trials with black cards; S_2, the stimulation present only on trials with grey cards; and S_c, the stimulation common to both types of trials. We denote by N_1, N_2, and N_c the number of cues in each of these subsets. In standard experiments the "cues" refer to experimentally manipulable aspects of the situation, such as tones, objects, colors, or symbols, and it is reasonably well known just how many different combinations of these cues will be responded to by subjects as distinct patterns. In some instances, however, the experimenter may have no a priori knowledge of the patterns distinguishable by the subject; in such instances the N_i may be treated as unknown parameters to be estimated from data, and the model may thus serve as a tool in securing evidence concerning the subject's perceptions of the physical situation.

Suppose, now, that the experimenter's procedure is to present on some trials (T_1-trials) a set of cues including m_1 from S_1 and m_c from S_c and on the remaining trials (T_2-trials) m_2 cues from S_2 and m_c from S_c. Further, let the two types of trials occur with equal frequencies in random sequence. On trials of type T_1 there will be $\binom{N_1}{m_1}\binom{N_c}{m_c}$ different patterns of cues available. Assuming that these patterns are all equally probable and letting $b_{1c} = \left[\binom{N_1}{m_1}\binom{N_c}{m_c}\right]^{-1}$, we can obtain an expression for probability of a correct response on a T_1-trial simply by appropriate substitution into Eq. 28, namely,

$$\Pr(A_{1,n_1} \mid T_{1,n_1}) = 1 - [1 - \Pr(A_{1,1} \mid T_{1,1})](1 - cb_{1c})^{n_1 - 1}, \quad (82)$$

where n_1 is the ordinal number of the T_1-trial. The corresponding function for T_2-trials is obtained similarly with parameter $b_{2c} = \left[\binom{N_2}{m_2}\binom{N_c}{m_c}\right]^{-1}$.

In the discrimination literature cues in the sets S_1 and S_2 are commonly referred to as *relevant* and those in S_c as *irrelevant*, since S_1 and S_2 are associated with reinforcing events, whereas the S_c are not. It is apparent by inspection of Eq. 82 that (for the foregoing specified experimental conditions) the pattern model predicts that probability of correct responding will go asymptotically to unity regardless of the numbers of relevant and irrelevant cues, provided only that neither m_1 nor m_2 is equal to zero. Rate of approach to asymptote on each type of trial is inversely related to the total number of patterns available for sampling. Therefore, other things being equal, rate of learning is decreased (and total errors to criterion increased) by the addition of either relevant or irrelevant cues.

Case 2. Only a subset of the cues presented on each trial is sampled. We consider now the situation that arises if the number of cues presented per trial is too large, or the exposure time too short, for the entire stimulus display to be sampled by the subject. Let us suppose that there are only two stimulus displays. The display on T_1-trials comprises the N_1 cues of S_1 together with the N_c cues of S_c, and that on T_2-trials, the N_2 cues of S_2 together with the N_c cues of S_c; further, to simplify the analysis let $N_1 = N_2 = N$. For a given fixed exposure time we assume a fixed sample size s, with all samples of exactly s cues being equiprobable. On T_1-trials, then, there will be $\binom{N}{s_1}\binom{N_c}{s - s_1}$ ways of filling the sample with s_1 cues from S_1 and the remainder from S_c. The asymptote of discriminative performance will depend on the size of s in relation to N_c. If $s \leqslant N_c$, so that the entire sample can come from the set of irrelevant cues, then the asymptotic probability of a correct response will be less than unity.

In Case 2 two types of patterns need to be distinguished for each type of trial. We can limit consideration to T_1-trials, since analogous arguments hold for T_2. There may be some patterns that include only cues from S_c and learning with respect to them will be on a simple random reinforcement schedule. The proportion of such patterns, w_c, is given by

$$w_c = \frac{\binom{N_c}{s}}{\binom{N + N_c}{s}},$$

which is equal to zero if $s > N_c$. If T_1- and T_2-trials have equal probabilities, then the probability, to be denoted V_n, that a pattern containing only cues from S_c will be conditioned to the A_1-response on trial n can be obtained from Eq. 28 by setting $\pi_{12} = \pi_{21} = \frac{1}{2}$:

$$\Pr(A_{1,n}) = V_n \quad \text{and} \quad \frac{c}{N} = \frac{cw_c}{\binom{N_c}{s}} = cb,$$

where

$$b = \binom{N + N_c}{s}^{-1},$$

that is,

$$V_n = \tfrac{1}{2} - (\tfrac{1}{2} - V_1)(1 - cb)^{n-1}. \tag{83}$$

The remaining patterns available on T_1-trials all contain at least one cue from S_1 and thus occur only on trials when response A_1 is reinforced. The probability, to be denoted U_n, that any one of these is conditioned to A_1 on trial n may be similarly obtained by rewriting Eq. 28, this time with $\pi_{12} = 0$, $\pi_{21} = 1$, $\Pr(A_{1,n}) = U_n$, and $c/N = \frac{1}{2}cb$, that is,

$$U_n = 1 - (1 - U_1)(1 - \tfrac{1}{2}cb)^{n-1}, \tag{84}$$

where the factor $\frac{1}{2}$ enters because these patterns are available for sampling on only one half of the trials.

Now, to obtain the probability of an A_1-response if a T_1-display is presented on trial n, we need only combine Eqs. 83 and 84, weighting each by the probability of the appropriate type of pattern, namely,

$$\begin{aligned}\Pr(A_{1,n} \mid T_{1,n}) &= (1 - w_c)U_n + w_c V_n \\ &= 1 - w_c + \tfrac{1}{2}w_c - (1 - w_c)(1 - U_1)(1 - \tfrac{1}{2}cb)^{n-1} \\ &\quad - w_c(\tfrac{1}{2} - V_1)(1 - cb)^{n-1},\end{aligned} \tag{85a}$$

which may be simplified, if $U_1 = V_1 = \frac{1}{2}$, to

$$\Pr(A_{1,n} \mid T_{1,n}) = 1 - \tfrac{1}{2}w_c - \tfrac{1}{2}(1 - w_c)(1 - \tfrac{1}{2}cb)^{n-1}. \tag{85b}$$

The resulting expression for probability of a correct response has a number of interesting general properties. The asymptote, as anticipated, depends in a simple way on w_c, the proportion of "irrelevant patterns." When $w_c = 0$, the asymptotic probability of a correct response is unity; when $w_c = 1$, the whole process reduces to simple random reinforcement. Between these extremes, asymptotic performance varies inversely with w_c, so that the terminal proportion of correct responses on either type of trial provides a simple estimate of this parameter from data. The slope parameter cb could then be estimated from total errors over a series of trials. As in Case 1, the rate of approach to asymptote proves to depend only on the conditioning parameters and total number of patterns available for sampling; thus it is a joint function of the total number of cues $N + N_c$ and the sample size s but does not depend on the relative proportions of relevant and irrelevant cues. The last result may seem implausible, but it should be noted that the result depends on the simplifying assumption of the pattern model that there are no transfer effects from learning on one pattern to performance on another pattern that has component cues in common with the first. The situation in this regard is different for the "mixed model" to be discussed next.

5.2 A Mixed Model

The pattern model may provide a relatively complete account of discrimination data in situations involving only distinct, readily discriminable patterns of stimulation, as, for example the "paired-comparison" experiment discussed in Sec. 2.3 or the verbal discrimination experiment treated by Bower (1962). Also, this model may account for some aspects of the data (e.g., asymptotic performance level, trials to criterion) even in discrimination experiments in which similarity, or communality, among stimuli is a major variable. But, to account for other aspects of the data in cases of the latter type, it is necessary to deal with transfer effects throughout the course of learning. The approach to this problem which we now wish to consider employs no new conceptual apparatus but simply a combination of ideas developed in preceding sections.

In the *mixed model* the conceptualization of the discriminative situation and the learning assumptions is exactly the same as that of the pattern model discussed in Sec. 5.1. The only change is in the response rule and that is altered in only one respect. As before, we assume that once a stimulus pattern has become conditioned to a response it will evoke that response on each subsequent occurrence (unless on some later trial the pattern becomes reconditioned to a different response, as, for example, during reversal of a discrimination). The new feature concerns patterns which have not yet become conditioned to any of the response alternatives of the given experimental situation but which have component cues in common with other patterns that have been so conditioned. Our assumption is simply that transfer occurs from a conditioned to an unconditioned pattern in accordance with the assumptions utilized in our earlier treatment of compounding and generalization (specifically, by axiom C2, together with a modified version of C1, of Sec. 3.1).

Before the assumptions about transfer can be employed unambiguously in connection with the mixed model, the notion of conditioned status of a component cue needs to be clarified. We shall say that a cue is conditioned to response A_i if it is a component of a stimulus pattern that has become conditioned to response A_i. If a cue belongs to two patterns, one of which is conditioned to response A_i and one to response A_j $(i \neq j)$, then the conditioning status of the cue follows that of the more recently conditioned pattern. If a cue belongs to no conditioned pattern, then it is said to be in the unconditioned, or "guessing," state. Note that a pattern may be unconditioned even though all of its cues are conditioned. Suppose for example, that a pattern consisting of cues x, y, and z in a particular arrangement has never been presented during the first n trials of an experiment but that each of the cues has appeared in other patterns, say wxy and wvz, which have been presented and conditioned. Then all of the cues of pattern xyz would be conditioned, but the pattern would still be in the unconditioned state. Consequently, if wxy had been conditioned to response A_1 and wvz to A_2, the probability of A_1 in the presence of pattern xyz would be $\frac{2}{3}$; but, if response A_1 were effectively reinforced in the presence of xyz, its probability of evocation by that pattern would henceforth be unity.

The only new complication arises if an unconditioned pattern includes cues that are still in the unconditioned state. Several alternative ways of formulating the response rule for this case have some plausibility, and it is by no means sure that any one choice will prove to hold for all types of situations. We shall limit consideration to the formulation suggested by a recent study of discrimination and transfer which has been analyzed in terms of the mixed model (Estes & Hopkins, 1961). The amended response rule is a direct generalization of Axiom C2 of Sec. 3.1; specifically, for a situation involving r response alternatives the following assumptions will apply:

1. If all cues in a pattern are unconditioned, the probability of any response A_i is equal to $1/r$.

2. If a pattern (sample) comprises m cues conditioned to response A_i, m' cues conditioned to other responses, and m'' unconditioned cues, then the probability that A_i will be evoked by this pattern is given by

$$\Pr(A_i) = \frac{m + (m''/r)}{m + m' + m''} .$$

In other words, Axiom C2 holds but with each unconditioned cue contributing "weight" $1/r$ toward the evocation of each of the alternative responses.

To illustrate these assumptions in operation, let us consider a simple classical discrimination experiment involving three cues, a, b, and c, and two responses, A_1 and A_2. We shall assume that the pattern ac is presented on half of the trials, with A_1 reinforced, and bc on the other half of the trials, with A_2 reinforced, the two types of trials occurring in random sequence. We assume further that conditions are such as to ensure the subject's sampling both cues presented on each trial. In a tabulation of the possible conditioning states of each pattern a 1, 2, or 0, respectively, in a state column indicates that the pattern is conditioned to A_1, conditioned to A_2, or unconditioned. For each pair of values under States, the associated

A_1-probabilities, computed according to the modified response rule, are given in the corresponding positions under A_1-probability. To reduce algebraic complications, we shall carry out derivations for the special case in which the subject starts the experiment with both patterns unconditioned. Then, under the conditions of reinforcement specified, only

States		A_1-Probability to Each Pattern	
ac	bc	ac	bc
1	2	1	0
1	1	1	1
2	2	0	0
2	1	0	1
0	1	$\frac{3}{4}$	1
0	2	$\frac{1}{4}$	0
1	0	1	$\frac{3}{4}$
2	0	0	$\frac{1}{4}$
0	0	$\frac{1}{2}$	$\frac{1}{2}$

the states represented in the first, seventh, sixth, and ninth rows of the table are available to the subject, and for brevity we number these states 3, 2, 1, and 0, in the order just listed; that is,

State 3 = pattern *ac* conditioned to A_1, and pattern *bc* conditioned to A_2.
State 2 = pattern *ac* conditioned to A_1, and pattern *bc* unconditioned.
State 1 = pattern *ac* unconditioned, and pattern *bc* conditioned to A_2.
State 0 = both patterns *ac* and *bc* are unconditioned.

Now, these states can be interpreted as the states of a Markov chain, since the probability of transition from any one of them to any other on a given trial is independent of the preceding history. The matrix of probabilities for one-step transitions among the four states takes the following form:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{c}{2} & 1-\frac{c}{2} & 0 & 0 \\ \frac{c}{2} & 0 & 1-\frac{c}{2} & 0 \\ 0 & \frac{c}{2} & \frac{c}{2} & 1-c \end{bmatrix}, \tag{86}$$

where the states are ordered 3, 2, 1, 0 from top to bottom and left to right. Thus State 3 (in which ac is conditioned to A_1 and bc to A_2) is an absorbing state, and the process must terminate in this state, with asymptotic probability of a correct response to each pattern equal to unity. In State 2 pattern ac is conditioned to A_1, but bc is still unconditioned. This state can be reached only from State 0, in which both patterns are unconditioned; the probability of the transition is $\frac{1}{2}$ (the probability that pattern ac will be presented) times c (the probability that the reinforcing event will produce conditioning); thus the entry in the second cell of the bottom row is $c/2$. From State 2 the subject can go only to State 3, and this transition again has probability $c/2$. The other cells are filled in similarly.

Now the probability $u_{i,n}$ of being in state i on trial n can be derived quite easily for each state. The subject is assumed to start the experiment in State 0 and has probability c of leaving this state on each trial; hence

$$u_{0,n} = (1 - c)^{n-1}.$$

For State 1 we can write a recursion,

$$u_{1,n} = \left(1 - \frac{c}{2}\right)^{n-2} \frac{c}{2} + \left(1 - \frac{c}{2}\right)^{n-3} (1 - c)\frac{c}{2} + \dots + (1 - c)^{n-2}\frac{c}{2},$$

which holds if $n \geqslant 2$. To be in State 1 on trial n the subject must have entered at the end of trial 1, which has probability $c/2$, and then remained for $n - 2$ trials, which has probability $[(1 - (c/2)]^{n-2}$; have entered at the end of trial 2, which has probability $(1 - c)(c/2)$, and then remained for $n - 3$ trials, which has probability $[1 - (c/2)]^{n-3}$; ... ; or have entered at the end of trial $n - 1$, which has probability $(1 - c)^{n-2}(c/2)$. The right-hand side of this recursion can be summed to yield

$$\begin{aligned} u_{1,n} &= \frac{c}{2}(1 - c)^{n-2} \sum_{v=0}^{n-2} \left[\frac{1 - (c/2)}{1 - c}\right]^{v} \\ &= (1 - c)^{n-1}\left\{\left[\frac{2 - c}{2(1 - c)}\right]^{n-1} - 1\right\} \\ &= \left(1 - \frac{c}{2}\right)^{n-1} - (1 - c)^{n-1}. \end{aligned}$$

By an identical argument we obtain

$$u_{2,n} = \left(1 - \frac{c}{2}\right)^{n-1} - (1 - c)^{n-1},$$

and then by subtraction

$$\begin{aligned} u_{3,n} &= 1 - u_{2,n} - u_{1,n} - u_{0,n} \\ &= 1 - 2\left(1 - \frac{c}{2}\right)^{n-1} + (1 - c)^{n-1}. \end{aligned}$$

From the tabulation of states and response probabilities we know that the probability of response A_1 to pattern ac is equal to 1, 1, $\frac{1}{4}$, and $\frac{1}{2}$, respectively, when the subject is in State 3, 2, 1, or 0. Consequently the probability of a correct (A_1) response to ac is obtained simply by summing these response probabilities, each weighted by the state probability, namely,

$$\begin{aligned}\Pr(A_{1,n} \mid ac) &= u_{3,n} + u_{2,n} + \frac{1}{4}u_{1,n} + \frac{1}{2}u_{0,n}\\ &= 1 - 2\left(1 - \frac{c}{2}\right)^{n-1} + (1-c)^{n-1} + \left(1 - \frac{c}{2}\right)^{n-1}\\ &\quad - (1-c)^{n-1} + \frac{1}{4}\left(1 - \frac{c}{2}\right)^{n-1} - \frac{1}{4}(1-c)^{n-1}\\ &\quad + \frac{1}{2}(1-c)^{n-1}\\ &= 1 - \frac{3}{4}\left(1 - \frac{c}{2}\right)^{n-1} + \frac{1}{4}(1-c)^{n-1}. \qquad (87)\end{aligned}$$

Equation 87 is written for the probability of an A_1-response to ac on trial n; however, the expression for probability of an A_2-response to bc is identical, and consequently Eq. 87 expresses also the probability p_n of a correct response on any trial, without regard to the stimulus pattern presented. A simple estimator of the conditioning parameter c is now obtainable by summing the error probability over trials. Letting e denote the expected total errors during learning, we have

$$\begin{aligned}e &= \sum_{n=1}^{\infty}(1 - p_n)\\ &= \frac{3}{4}\sum_{n=1}^{\infty}\left(1 - \frac{c}{2}\right)^{n-1} - \frac{1}{4}\sum_{n=1}^{\infty}(1-c)^{n-1}\\ &= \frac{3}{4}\frac{2}{c} - \frac{1}{4}\frac{1}{c}\\ &= \frac{5}{4c}.\end{aligned}$$

An example of the sort of prediction involving a relatively direct assessment of transfer effects is the following. Suppose the first stimulus pattern to appear is ac; the probability of a correct response to it is, by hypothesis, $\frac{1}{2}$, and if there were no transfer between patterns the probability of a correct response to bc when it first appeared on a later trial should be $\frac{1}{2}$ also. Under the assumptions of the mixed model, however, the probability of a correct response to bc, if it first appeared on trial 2, should be

$$\frac{[1 - \frac{1}{2}(1 - c) - c] + \frac{1}{2}}{2} = \frac{1}{2} - \frac{c}{4};$$

if it first appeared on trial 3, it should be

$$\frac{\frac{1}{2}(1 - c)^2 + \frac{1}{2}}{2} = \frac{1}{2} - \frac{c}{2}\left(1 - \frac{c}{2}\right);$$

and so on, tending to $\frac{1}{4}$ after a sufficiently long prior sequence of ac trials.

Simply by inspection of the transition matrix we can develop an interesting prediction concerning behavior during the presolution period of the experiment. By presolution period we mean the sequence of trials before the last error for any given subject. We know that the subject cannot be in State 3 on any trial before the last error. On all trials of the presolution period the probability of a correct response should be equal either to $\frac{1}{2}$ (if no conditioning has occurred) or to $\frac{5}{8}$ (if exactly one of the two stimulus patterns has been conditioned to its correct response). Thus the proportion, which we denote by P_{ps}, of correct responses over the presolution trial sequence should fall in the interval

$$\tfrac{1}{2} \leqslant P_{ps} \leqslant \tfrac{5}{8},$$

and, in fact, the same bounds obtain for any subset of trials within the presolution sequence. Clearly, predictions from this model concerning presolution responding differ sharply from those derivable from any model that assumes a continuous increase in probability of correct responding during the presolution period; this model also differs, though not so sharply, from a pure "insight" model that assumes no learning on presolution trials. As far as we know, no data relevant to these differential predictions are available in the literature (though similar predictions have been tested in somewhat different situations: Suppes & Ginsberg, 1963; Theios, 1963). Now that the predictions are in hand, it seems likely that pertinent analyses will be forthcoming.

The development in this section was for the case in which there were only three cues, a, b, and c. For the more general case we could assume that there are N_a cues associated with stimulus a, N_b with stimulus b, and N_c with stimulus c. If we assume, as we have in this section, that experimental conditions are such to ensure the subject's sampling all cues presented on each trial, then Eq. 87 may be rewritten as

$$\Pr(A_{1,n} \mid ac) = 1 - \frac{1}{2}(1 + w_1)\left(1 - \frac{c}{2}\right)^{n-1} + \frac{1}{2}w_1(1 - c)^{n-1}$$

$$\Pr(A_{2,n} \mid bc) = 1 - \frac{1}{2}(1 + w_2)\left(1 - \frac{c}{2}\right)^{n-1} + \frac{1}{2}w_2(1 - c)^{n-1},$$

where

$$w_1 = \frac{N_c}{N_a + N_c} \quad \text{and} \quad w_2 = \frac{N_c}{N_b + N_c}.$$

Further,

$$e = \sum_{n=1}^{\infty} \left\{ \frac{1}{2} [1 - \Pr(A_{1,n} \mid ac)] + \frac{1}{2} [1 - \Pr(A_{2,n} \mid bc)] \right\}$$
$$= \frac{1}{c}\left(1 + \frac{1}{2}\bar{w}\right),$$

where $\bar{w} = \frac{1}{2}(w_1 + w_2)$. The parameter $\bar{w}$ is an index of similarity between the stimuli ac and bc; as $\bar{w}$ approaches its maximum value of 1, the number of total errors increases. Further, the proportion of correct responses over the presolution trial sequence should fall in the interval

$$\tfrac{1}{2} \leqslant P_{ps} \leqslant \tfrac{1}{2} + \tfrac{1}{4}(1 - w_1)$$

or in the interval

$$\tfrac{1}{2} \leqslant P_{ps} \leqslant \tfrac{1}{2} + \tfrac{1}{4}(1 - w_2),$$

depending on whether ac or bc is conditioned first.

5.3 Component Models

As long as the number of stimulus patterns involved in a discrimination experiment is relatively small, an analysis in terms of an appropriate case of the mixed model can be effected along the lines indicated in Sec. 5.2. But the number of cues need become only moderately large in order to generate a number of patterns so great as to be unmanageable by these methods. However, if the number of patterns is large enough so that any particular pattern is unlikely to be sampled more than once during an experiment, the emendations of the response rule presented in Sec. 5.2 can be neglected and the process treated as a simple extension of the component model of Sec. 4.1.

Suppose, for example, that a classical discrimination involved a set, S_1, of cues available only on trials when A_1 is reinforced, a set, S_2, of cues available only on trials when A_2 is reinforced, and a set, S_c, of cues available on all trials; further, assume that a constant fraction of each set presented is sampled by the subject on any trial. If the two types of trials occur with equal probabilities and if the numbers of cues in the various sets are large enough so that the number of possible trial samples is larger than the number of trials in the experiment, then we may apply Eq. 53 of Sec. 3.3 to obtain approximate expressions for response probabilities. For example, asymptotically all of the N_1 elements of S_1 and half of the N_c elements of S_c

(on the average) would be conditioned to response A_1, and therefore probability of A_1 on a trial when S_1 was presented would be predicted by the component model to be

$$\Pr(A_1 \mid S_1) = \frac{N_1 + \frac{1}{2}N_c}{N_1 + N_c},$$

which will, in general, have a value intermediate between $\frac{1}{2}$ and unity. Functions for learning curves and other aspects of the data can be derived for various types of discrimination experiments from the assumptions of the component model. Numerous results of this sort have been published (Burke & Estes, 1957; Bush & Mosteller, 1951b; Estes, 1958, 1961a; Estes, Burke, Atkinson & Frankmann, 1957; Popper, 1959; Popper & Atkinson, 1958).

5.4 Analysis of a Signal Detection Experiment

Although, so far, we have developed stimulus sampling models only in connection with simple associative learning and discrimination learning, it should be noted that such models may have much broader areas of application. On occasion we may even see possibilities of using the concepts of stimulus sampling and association to interpret experiments that, by conventional classifications, do not fall within the area of learning. In this section we examine such a case.

The experiment to be considered fits one of the standard paradigms associated with studies of signal detection (see, e.g., Tanner & Swets, 1954; Swets, Tanner, & Birdsall, 1961; or Chapter 3, Vol. 1, by Luce). The subject's task in this experiment, like that of an observer monitoring a radar screen, is to detect the presence of a visual signal which may occur from time to time in one of several possible locations. Problems of interest in connection with theories of signal detection arise when the signals are faint enough so that the observer is unable to report them with complete accuracy on all occasions. One empirical relation that we would want to account for, in quantitative detail, is that between detection probabilities and the relative frequencies with which signals occur in different locations. Another is the improvement in detection rate that may occur over a series of trials even when the observer receives no knowledge of results.

A possible way of accounting for the "practice effect" is suggested by some rather obvious analogies between the detection experiment and the probability learning experiment considered earlier: we expect that, when the subject actually detects a signal (in terms of stimulus sampling theory, samples the corresponding stimulus element), he will make the appropriate verbal report. Further, in the absence of any other information, this detection of the signal may act as a reinforcing event, leading to conditioning of the verbal report to other cues in the situation which may have been available for sampling before the occurrence of the signal. If so, and if signals occur in some locations more often than in others, then on the basis of the theory developed in earlier sections we should predict that the subject will come to report the signal in the preferred location more frequently than in others on trials when he fails to detect a signal and is forced to respond to background cues. These notions are made more explicit in connection with the following analysis of a visual recognition experiment reported by Kinchla (1962).

Kinchla employed a forced-choice, visual-detection situation involving a series of more than 900 discrete trials for each subject. Two areas were outlined on a uniformly illuminated milk-glass screen. Each trial began with an auditory signal, during which one of the following events occurred:

1. A fixed increment in radiant intensity occurred in area 1—a T_1-type trial.
2. A fixed increment in radiant intensity occurred in area 2—a T_2-type trial.
3. No change in the radiant character of either signal area occurred—a T_0-type trial.

Subjects were told that a change in illumination would occur in one of the two areas on each trial. Following the auditory signal, the subject was required to make either an A_1- or an A_2-response (i.e., select one of two keys placed below the signal area) to indicate the area he believed had changed in brightness. The subject was given no information at the end of the trial as to whether his response was correct. Thus, on a given trial, one of three events occurred (T_1, T_2, T_0), the subject made either an A_1- or an A_2-response, and a short time later the next trial began.

For a fixed signal intensity, the experimenter has the option of specifying a schedule for presenting the T_i-events. Kinchla selected a simple probabilistic procedure in which $\Pr(T_{i,n}) = \xi_i$ and $\xi_1 + \xi_2 + \xi_0 = 1$. Two groups of subjects were run. For Group *I*, $\xi_1 + \xi_2 = 0.4$ and $\xi_0 = 0.2$. For Group II, $\xi_1 = \xi_0 = 0.2$ and $\xi_2 = 0.6$. The purpose of Kinchla's study was to determine how these event schedules influenced the likelihood of correct detections.

The model that we shall use to analyze the experiment combines two quite distinct processes: a simple perceptual process defined with regard to the signal events and a learning process associated with background cues. The stimulus situation is conceptually represented in terms of two *sensory elements*, s_1 and s_2, corresponding to the two alternative signals, and a set, S, of elements associated with stimulus features common to all trials. On every trial the subject is assumed to sample a single element from the background set S, and he may or may not sample one of the sensory elements. If the s_1 element is sampled, an A_1 occurs; if s_2 is sampled, an A_2 occurs. If neither sensory element is sampled, the subject makes the response to which the background element is conditioned. Conditioning of elements in S changes from trial to trial via a learning process.

The sampling of sensory elements depends on the trial type (T_1, T_2, T_0) and is described by a simple probabilistic model. The learning process associated with S is assumed to be the multi-element pattern model presented in Sec. 2. Specifically, the assumptions of the model are embodied in the following statements:

1. If T_i ($i = 1, 2$) occurs, then sensory element s_i will be sampled with probability h (with probability $1 - h$ *neither* s_1 nor s_2 will be sampled). If T_0 occurs, then neither s_1 nor s_2 will be sampled.
2. Exactly one element is sampled from S on *every* trial. Given the set S of N elements, the probability of sampling a particular element is $1/N$.
3. If s_i ($i = 1, 2$) is sampled on trial n, then with probability c' the element sampled from S on the trial becomes conditioned to A_i at the end of trial n. If neither s_1 nor s_2 is sampled, then with probability c the element sampled from S becomes conditioned with equal likelihood to A_1 or A_2 at the end of trial n.
4. If sensory element s_i is sampled, then A_i will occur. If neither sensory element is sampled, then the response to which the sampled element from S is conditioned will occur.

If we let p_n denote the expected proportion of elements in S conditioned to A_1 at the start of trial n, then (in terms of statements 1 and 4) we can immediately write an expression for the likelihood of an A_i-response, given a T_j-event, namely,

$$\Pr(A_{1,n} \mid T_{1,n}) = h + (1 - h)p_n, \tag{88a}$$

$$\Pr(A_{2,n} \mid T_{2,n}) = h + (1 - h)(1 - p_n), \tag{88b}$$

$$\Pr(A_{1,n} \mid T_{0,n}) = p_n. \tag{88c}$$

The expression for p_n can be obtained from Statements 2 and 3 by the same methods used throughout Sec. 2 of this chapter (for a derivation of this result, see Atkinson, 1963a):

$$p_n = p_\infty - (p_\infty - p_1)\left[1 - \frac{1}{N}(a + b)\right]^{n-1},$$

where $a = \xi_1 hc' + (1 - h)(c/2) + \xi_0 h(c/2)$, $b = \xi_2 hc' + (1 - h)(c/2) + \xi_0 h(c/2)$, and $p_\infty = a/(a + b)$. Division of the numerator and denominator of p_∞ by c yields the expression

$$p_\infty = \frac{\xi_1 h\psi + \frac{1}{2}(1 - h) + \xi_0 h\frac{1}{2}}{(1 - \xi_0)(1 - h + h\psi) + \xi_0}, \tag{89}$$

where $\psi = c'/c$. Thus the asymptotic expression for p_n does not depend on the absolute values of c' and c but only on their ratio.

An inspection of Kinchla's data indicates that the curves for $\Pr(A_i \mid T_j)$ are extremely stable over the last 400 or so trials of the experiment; consequently we shall view this portion of the data as asymptotic. Table 7

Table 7 Predicted and Observed Asymptotic Response Probabilities for Visual Detection Experiment

	Group I		Group II	
	Observed	Predicted	Observed	Predicted
$\Pr(A_1 \mid T_1)$	0.645	0.645	0.558	0.565
$\Pr(A_2 \mid T_2)$	0.643	0.645	0.730	0.724
$\Pr(A_1 \mid T_0)$	0.494	0.500	0.388	0.388

presents the observed mean values of $\Pr(A_i \mid T_j)$ for the last 400 trials. The corresponding asymptotic expressions are specified in terms of Eqs. 88 and 89 and are simply

$$\lim_{n\to\infty} \Pr(A_{1,n} \mid T_{1,n}) = h + (1 - h)p_\infty, \tag{90a}$$

$$\lim_{n\to\infty} \Pr(A_{2,n} \mid T_{2,n}) = h + (1 - h)(1 - p_\infty), \tag{90b}$$

$$\lim_{n\to\infty} \Pr(A_{1,n} \mid T_{0,n}) = p_\infty. \tag{90c}$$

In order to generate asymptotic predictions, we need values for h and ψ. We first note by inspection of Eq. 89 that $p_\infty = \frac{1}{2}$ for Group I; in fact, whenever $\xi_1 = \xi_2$, we have $p_\infty = \frac{1}{2}$. Hence taking the observed asymptotic value for $\Pr(A_1 \mid T_1)$ in Group I (i.e., 0.645) and setting it equal to $h + (1 - h)\frac{1}{2}$ yields an estimate of $h = 0.289$. The background illumination and the increment in radiant intensity are the same for both experimental groups, and therefore we would require an estimate of h obtained from Group I to be applicable to Group II. In order to estimate ψ, we take the observed asymptotic value of $\Pr(A_1 \mid T_0)$ in Group II and set it equal to the right side of Eq. 89 with $h = 0.289$, $\xi_1 = \xi_0 = 0.2$, and $\xi_2 = 0.6$; solving for ψ, we obtain $\hat{\psi} = 2.8$. Use of these estimates of h and ψ in Eqs. 89 and 90 yields the asymptotic predictions given in Table 7.

Over-all, the equations give an excellent account of these particular response measures. However, a more crucial test of the model is provided by an analysis of the sequential data. To indicate the nature of the sequential predictions that can be obtained, consider the probability of an A_1-response on a T_1-trial, given the various trial types and responses that can occur on the preceding trial, that is,

$$\Pr(A_{1,n+1} \mid T_{1,n+1}A_{i,n}T_{j,n}),$$

where $i = 1, 2$ and $j = 0, 1, 2$. Explicit expressions for these quantities can be derived from the axioms by the same methods used throughout this chapter. To indicate their form, theoretical expressions for

$$\lim_{n\to\infty} \Pr(A_{1,n+1} \mid T_{1,n+1}A_{i,n}T_{j,n})$$

are given, and, to simplify notation, they are written as $\Pr(A_1 \mid T_1A_iT_j)$. The expressions for these quantities are as follows:

$$\Pr(A_1 \mid T_1A_1T_1) = \frac{[h + (1-h)\,\delta]p_\infty + (1-p_\infty)h\gamma'}{NX} + \frac{(N-1)X}{N}, \tag{91a}$$

$$\Pr(A_1 \mid T_1A_2T_1) = \frac{(1-h)\,\delta'(1-p_\infty)}{N(1-X)} + \frac{(N-1)X}{N}, \tag{91b}$$

$$\Pr(A_1 \mid T_1A_2T_2) = \frac{h\gamma p_\infty + [h^2 + (1-h)\,\delta'](1-p_\infty)}{NY} + \frac{(N-1)X}{N}, \tag{91c}$$

$$\Pr(A_1 \mid T_1A_1T_2) = \frac{(1-h)\,\delta p_\infty}{N(1-Y)} + \frac{(N-1)X}{N}, \tag{91d}$$

$$\Pr(A_1 \mid T_1A_1T_0) = \frac{\delta}{N} + \frac{(N-1)X}{N}, \tag{91e}$$

$$\Pr(A_1 \mid T_1A_2T_0) = \frac{\delta'}{N} + \frac{(N-1)X}{N}, \tag{91f}$$

where

$$\gamma = c'h + (1-c'),$$
$$\gamma' = c' + (1-c')h,$$
$$\delta = (c/2)h + [1-(c/2)],$$
$$\delta' = (c/2) + [1-(c/2)]h,$$
$$X = h + (1-h)p_\infty,$$

and

$$Y = h + (1-h)(1-p_\infty).$$

It is interesting to note that the asymptotic expressions for $\Pr(A_{1,n} \mid T_{j,n})$ depend only on h and ψ, whereas the quantities in Eq. 91 are functions of all four parameters N, c, c', and h. Comparable sets of equations can be written for $\Pr(A_2 \mid T_2A_iT_j)$ and $\Pr(A_1 \mid T_0A_iT_j)$.

The expressions in Eq. 91 are rather formidable, but numerical predictions can be easily calculated once values for the parameters have been obtained. Further, independent of the parameter values, certain relations among the sequential probabilities can be specified. As an example of such

Table 8 Predicted and Observed Asymptotic Sequential Response Probabilities in Visual-Detection Experiment

	Group I		Group II	
	Observed	Predicted	Observed	Predicted
$\Pr(A_2 \mid T_2A_1T_1)$	0.57	0.58	0.59	0.64
$\Pr(A_2 \mid T_2A_2T_1)$	0.65	0.69	0.70	0.76
$\Pr(A_2 \mid T_2A_2T_2)$	0.71	0.71	0.79	0.77
$\Pr(A_2 \mid T_2A_1T_2)$	0.61	0.59	0.69	0.66
$\Pr(A_2 \mid T_2A_1T_0)$	0.54	0.59	0.68	0.66
$\Pr(A_2 \mid T_2A_2T_0)$	0.66	0.70	0.71	0.76
$\Pr(A_1 \mid T_1A_1T_1)$	0.73	0.71	0.70	0.65
$\Pr(A_1 \mid T_1A_2T_1)$	0.62	0.59	0.59	0.52
$\Pr(A_1 \mid T_1A_2T_2)$	0.53	0.58	0.53	0.51
$\Pr(A_1 \mid T_1A_1T_2)$	0.66	0.70	0.64	0.64
$\Pr(A_1 \mid T_1A_1T_0)$	0.72	0.70	0.61	0.63
$\Pr(A_1 \mid T_1A_2T_0)$	0.61	0.59	0.48	0.52
$\Pr(A_2 \mid T_0A_1T_1)$	0.38	0.40	0.47	0.49
$\Pr(A_2 \mid T_0A_2T_1)$	0.56	0.58	0.59	0.66
$\Pr(A_2 \mid T_0A_2T_2)$	0.64	0.60	0.67	0.68
$\Pr(A_2 \mid T_0A_1T_2)$	0.47	0.42	0.51	0.51
$\Pr(A_2 \mid T_0A_1T_0)$	0.47	0.42	0.50	0.51
$\Pr(A_2 \mid T_0A_2T_0)$	0.60	0.58	0.65	0.66

a relation, it can be shown that $\Pr(A_1 \mid T_1A_1T_0) \geqslant \Pr(A_1 \mid T_1A_2T_0)$ for any stimulus schedule and any set of parameter values. To see this, simply subtract Eq. 91f from Eq. 91e and note that $\delta \geqslant \delta'$.

In Table 8 the observed values for $\Pr(A_i \mid T_jA_kT_l)$ are presented as reported by Kinchla. Estimates of these conditional probabilities were computed for individual subjects, using the data over the last 400 trials; the averages of these individual estimates are the quantities given in the table. Each entry is based on 24 subjects.

In order to generate theoretical predictions for the observed entries in Table 8, values for N, c, c', and h are needed. Of course, estimates of h and $\psi = c'/c$ have already been made for this set of data, and therefore it is necessary only to estimate N and either c or c'. We obtain our estimates of N and c by a least-squares method; that is, we select a value of N and c (where $c' = c\psi$) so that the sum of squared deviations between the 36 observed values in Table 8 and the corresponding theoretical quantities is minimized. The theoretical quantities for $\Pr(A_1 \mid T_1A_iT_j)$ are computed from Eq. 91; theoretical expressions for $\Pr(A_2 \mid T_2A_iT_j)$ and $\Pr(A_2 \mid T_0A_iT_j)$ have not been presented here but are of the same general form as those given in Eq. 91.

With this technique, estimates of the parameters are as follows:

$$\begin{aligned} N &= 4.23 \qquad & c' &= 1.00 \\ h &= 0.289 \qquad & c &= 0.357. \end{aligned} \tag{92}$$

The predictions corresponding to these parameter values are presented in Table 8. When we note that only four of the possible 36 degrees of freedom represented in Table 8 have been utilized in estimating parameters, the close correspondence between theoretical and observed quantities may be interpreted as giving considerable support to the assumptions of the model.

A great deal of research needs to be done to explore the consequences of this approach to signal detection. In terms of the experimental problem considered in this section, much progress can be made via differential tests among alternative formulations of the model. For example, we postulated a multi-element pattern model to describe the learning process associated with background stimuli; it would be important to determine whether other formulations of the learning process such as those developed in Sec. 4 or those proposed by Bush and Mosteller (1955) would provide as good or even better theoretical fits than the ones displayed in Tables 7 and 8. Also, it would be valuable to examine variations in the scheme for sampling sensory elements along lines developed by Luce (1959, 1963) and Restle (1961).

More generally, further development of the theory is required before we can attempt to deal with the wide range of empirical phenomena encompassed in the approach to perception via decision theory proposed by Swets, Tanner, and Birdsall (1961) and others. Some theoretical work has been done by Atkinson (1963b) along the lines outlined in this section to account for the *ROC* (receiver-operating-characteristic) curves that are typically observed in detection studies and to specify the relation between forced-choice and yes-no experiments. However, this work is still quite tentative, and an evaluation of the approach will require extensive analyses of the detailed sequential properties of psychophysical data.

5.5 Multiple-Process Models

Analyses of certain behavioral situations have proved to require formulations in terms of two or more distinguishable, though possibly interdependent, learning processes that proceed simultaneously. For some situations these separate processes may be directly observable; for other situations we may find it advantageous to postulate processes that are unobservable but that determine in some well-defined fashion the sequence of observable behaviors. For example, in Restle's (1955) treatment of discrimination learning it is assumed that irrelevant stimuli may become "adapted" over a period of time and thus be rendered nonfunctional. Such an analysis entails a two-process system. One process has to do with the conditioning of stimuli to responses, whereas the other prescribes both the conditions under which cues become irrelevant and the rate at which adaptation occurs.

Another application of multiple-process models arises with regard to discrimination problems in which either a covert or a directly observable orienting response is required. One process might describe how the stimuli presented to the subject become conditioned to discriminative responses. Another might specify the acquisition and extinction of various orienting responses; these orienting responses would determine the specific subset of the environment that the subject would perceive on a given trial. For models dealing with this type of problem, see Atkinson (1958), Bush & Mosteller (1951b), Bower (1959), and Wyckoff (1952).

As another example, consider a two-process scheme developed by Atkinson (1960) to account for certain types of discrimination behavior. This model makes use of the distinction, developed in Secs. 2 and 3, between component models and pattern models and suggests that the subject may (at any instant in time) perceive the stimulus situation either as a unit pattern or as a collection of individual components. Thus two perceptual states are defined: one in which the subject responds to the pattern of stimulation and one in which he responds to the separate components of the situation. Two learning processes are also defined. One process specifies how the patterns and components become conditioned to responses, and the second process describes the conditions under which the subject shifts from one perceptual state to another. The control of the second process is governed by the reinforcing schedule, the subject's sequence of responses, and by similarity of the discriminanda. In this model neither the conditioning states nor the perceptual states are observable; nevertheless, the behavior of the subject is rigorously defined in terms of these hypothetical states.

Models of the sort described are generally difficult to work with mathematically and consequently have had only limited development and analysis. It is for this reason that we select a particularly simple example to illustrate the type of formulation that is possible. The example deals with a discrimination-learning task investigated by Atkinson (1961) in which observing responses are categorized and directly measured.

The experimental situation consists of a sequence of discrete trials. Each trial is specified in terms of the following classifications:

T_1, T_2: *Trial type.* Each trial is either a T_1 or a T_2. The trial type is set by the experimenter and determines *in part* the stimulus event occurring on the trial.

R_1, R_2: *Observing responses.* On each trial the subject makes either an R_1 or R_2. The particular observing response determines in part the stimulus event for that trial.

s_1, s_b, s_2: *Stimulus events.* Following the observing response, one and only one of these stimulus events (discriminative cues) occurs. On a T_1-trial either s_1 or s_b can occur; on a T_2-trial either s_2 or s_b can occur.[15]

A_1, A_2: *Discriminative responses.* On each trial the subject makes either an A_1- or A_2-response to the presentation of a stimulus event.

O_1, O_2: *Trial outcome.* Each trial is terminated with the occurrence of one of these events. An O_1 indicates that A_1 was the correct response for that trial and O_2 indicates that A_2 was correct.

The sequence of events on a trial is as follows: (1) The ready signal occurs and the subject responds with R_1 or R_2. (2) Following the observing response, s_1, s_2, or s_b is presented. (3) To the onset of the stimulus event the subject responds with either A_1 or A_2. (4) The trial terminates with either an O_1- or O_2-event.

To keep the analysis simple, we consider an experimenter-controlled reinforcement schedule. On a T_1-trial either an O_1 occurs with probability π_1 or an O_2 with probability $1 - \pi_1$; on a T_2-trial an O_1 occurs with probability π_2 or an O_2 with probability $1 - \pi_2$. The T_1-trial occurs with probability β and T_2 with probability $1 - \beta$. Thus a T_1-O_1-combination occurs with probability $\beta\pi_1$, a T_1-O_2, with probability $\beta(1 - \pi_1)$, and so on.

The particular stimulus event s_i $(i = 1, 2, b)$ that the experimenter

[15] The subscript b has been used to denote the stimulus event that may occur on *both* T_1- and T_2-trials; the subscripts 1 and 2 denote stimulus events unique to T_1- and T_2-trials, respectively.

presents on any trial depends on the trial type (T_1 or T_2) and the subject's observing response (R_1 or R_2).

1. If an R_1 is made, then
   (*a*) with probability α the s_1-event occurs on a T_1-trial and the s_2-event on a T_2-trial;
   (*b*) with probability $1 - \alpha$ the s_b-event occurs, regardless of the trial type.
2. If an R_2 is made, then
   (*a*) with probability α the s_b-event occurs, regardless of the trial type;
   (*b*) with probability $1 - \alpha$ the s_1-event occurs on a T_1-trial and s_2 on a T_2-trial.

To clarify this procedure, consider the case in which $\alpha = 1$, $\pi_1 = 1$, and $\pi_2 = 0$. If the subject is to be correct on every trial, he must make an A_1 on a T_1-trial and an A_2 on a T_2-trial. However, the subject can ascertain the trial type only by making the appropriate observing response; that is, R_1 must be made in order to identify the trial type, for the occurrence of R_2 always leads to the presentation of s_b, regardless of the trial type. Hence for perfect responding the subject must make R_1 with probability 1 and then make A_1 to s_1 or A_2 to s_2. The purpose of the Atkinson study was to determine how variations in π_1, π_2, and α would affect both the observing responses and the discriminative responses.

Our analysis of this experimental procedure is based on the axioms presented in Secs. 1 and 2. However, in order to apply the theory, we must first identify the stimulus and reinforcing events in terms of the experimental operations. The identification we offer seems quite natural to us and is in accord with the formulations given in Secs. 1 and 2.

We assume that associated with the ready signal is a set S_R of pattern elements. Each element in S_R is conditioned to the R_1- or the R_2-observing response; there are N' such elements. At the start of each trial (i.e., with the onset of the ready signal) an element is sampled from S_R, and the subject makes the response to which the element is conditioned.

Associated with each stimulus event, s_i ($i = 1, 2, b$), is a set, S_i, of pattern elements; elements in S_i are conditioned to the A_1- or the A_2-discriminative response. There are N such elements in each set, S_i, and for simplicity we assume that the sets are pairwise disjoint. When the stimulus event s_i occurs, one element is randomly sampled from S_i, and the subject makes the discriminative response to which the element is conditioned.

Thus we have two types of learning processes: one defined on the set S_R and the other defined on the sets S_1, S_b, and S_2. Once the reinforcing events have been specified for these processes, we can apply our axioms. The interpretation of reinforcement for the discriminative-response process is identical to that given in Sec. 2. If a pattern element is sampled from set S_i for $i = 1, 2, b$ and is followed by an O_j outcome, then with probability c the element becomes conditioned to A_j and with probability $1 - c$ the conditioning state of the sampled element remains unchanged.

The conditioning process for the S_R set is somewhat more complex in that the reinforcing events for the observing responses are assumed to be subject-controlled. Specifically, if an element conditioned to R_i is sampled from S_R and followed by either an A_1O_1- or A_2O_2-event, then the element will remain conditioned to R_i; however, if A_1O_2 or A_2O_1 occurs, then with probability c' the element will become conditioned to the *other* observing response. Otherwise stated, if an element from S_R elicits an observing response that selects a stimulus event and, in turn, the stimulus event elicits a correct discriminative response (i.e., A_1O_1 or A_2O_2), then the sampled element will remain conditioned to that observing response. However, if the observing response selects a stimulus event that gives rise to an incorrect discriminative response (i.e., A_1O_2 or A_2O_1), then there will be a decrement in the tendency to repeat that observing response on the next trial.

Given the foregoing identification of events, we can now generate a mathematical model for the experiment. To simplify the analysis, we let $N' = N = 1$; namely, we assume that there is one element in each of our stimulus sets and consequently the single element is sampled with probability 1 whenever the set is available. With this restriction we may describe the conditioning state of a subject at the start of each trial by an ordered four-tuple $\langle ijkl \rangle$:

1. The first member i is 1 or 2 and indicates whether the single element of S_R is conditioned to R_1 or R_2.
2. The second member j is 1 or 2 and indicates whether the single element of S_1 is conditioned to A_1 or A_2.
3. The third member k is 1 or 2 and indicates whether the element of S_b is conditioned to A_1 or A_2.
4. The fourth member l is 1 or 2 and indicates whether the element of S_2 is conditioned to A_1 or A_2.

Thus, if the subject is in state $\langle ijkl \rangle$, he will make the R_i observing response; then, to s_1, s_b, or s_2, he will make discriminative response A_j, A_k, or A_l, respectively.

From our assumptions it follows that the sequence of random variables that take the subject states $\langle ijkl \rangle$ as values is a 16-state Markov chain.

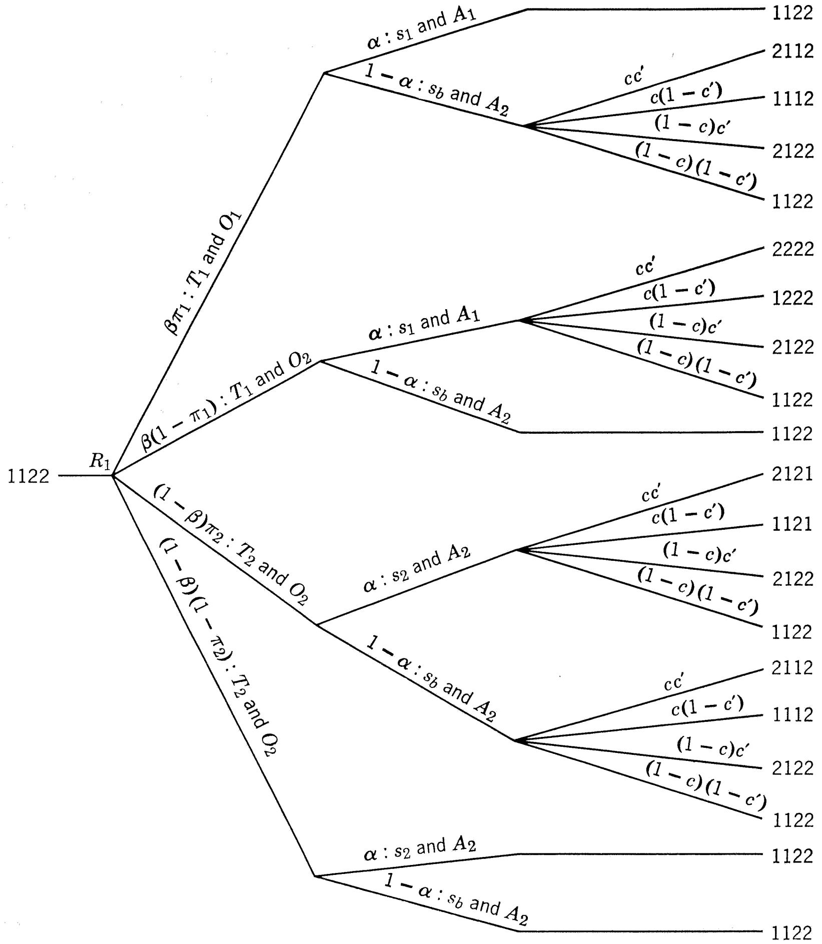

Fig. 10. Branching process, starting in state ⟨1122⟩, for a single trial in the two-process discrimination-learning model.

Figure 10 displays the possible transitions that can occur when the subject is in state ⟨1122⟩ on trial n. To clarify this tree, let us trace out the top branch. An R_1 is elicited with probability 1, and with probability $\beta\pi_1$ a T_1-trial with an O_1-outcome will occur; further, given an R_1-response on a T_1-trial, there is probability α that the s_1-stimulus event will occur; the onset of the s_1-event elicits a correct response, hence no change occurs in the conditioning state of any of the stimulus patterns. Now consider the next set of branches: an R_1 occurs and we have a T_1O_1-trial; with probability $1 - \alpha$ the s_b-stimulus will be presented and an A_2 will occur; the A_2-response is incorrect (in that it is followed by an O_1-event); hence with probability c the element of set S_b will become conditioned to A_1 and with independent probability c' the element of set S_R will become conditioned to the alternative observing response, namely R_2.

From this tree we obtain probabilities corresponding to the $\langle 1122 \rangle$ row in the transition matrix. For example, the probability of going from $\langle 1122 \rangle$ to $\langle 2112 \rangle$ is simply $\beta\pi_1(1 - \alpha)cc' + (1 - \beta)\pi_2(1 - \alpha)cc'$; that is, the sum over branches 2 and 15. An inspection of the transition matrix yields important results. For example, if $\alpha = 1$, $\pi_1 = 1$, and $\pi_2 = 0$, then states $\langle 1112 \rangle$ and $\langle 1122 \rangle$ are absorbing, hence in the limit $\Pr(R_{1,n}) = 1$, $\Pr(A_{1,n} \mid T_{1,n}) = 1$, and $\Pr(A_{2,n} \mid T_{2,n}) = 1$.

As before, let $u_{ijkl}^{(n)}$ denote the probability of being in state $\langle ijkl \rangle$ on trial n; when the limit exists, let $u_{ijkl} = \lim_{n\to\infty} u_{ijkl}^{(n)}$. Experimentally, we are interested in evaluating the following theoretical predictions:

$$\begin{aligned}\Pr(R_{1,n}) = {}& u_{1111}^{(n)} + u_{1112}^{(n)} + u_{1121}^{(n)} + u_{1122}^{(n)} \\ & + u_{1211}^{(n)} + u_{1212}^{(n)} + u_{1221}^{(n)} + u_{1222}^{(n)},\end{aligned} \tag{93a}$$

$$\begin{aligned}\Pr(A_{1,n} \mid T_{1,n}) = {}& u_{1111}^{(n)} + u_{1112}^{(n)} + u_{2111}^{(n)} + u_{2112}^{(n)} \\ & + \alpha[u_{1121}^{(n)} + u_{1122}^{(n)} + u_{2211}^{(n)} + u_{2212}^{(n)}] \\ & + (1 - \alpha)[u_{1211}^{(n)} + u_{1212}^{(n)} + u_{2121}^{(n)} + u_{2122}^{(n)}],\end{aligned} \tag{93b}$$

$$\begin{aligned}\Pr(A_{1,n} \mid T_{2,n}) = {}& u_{1111}^{(n)} + u_{1211}^{(n)} + u_{2111}^{(n)} + u_{2211}^{(n)} \\ & + \alpha[u_{1121}^{(n)} + u_{1221}^{(n)} + u_{2112}^{(n)} + u_{2212}^{(n)}] \\ & + (1 - \alpha)[u_{1112}^{(n)} + u_{1212}^{(n)} + u_{2121}^{(n)} + u_{2221}^{(n)}],\end{aligned} \tag{93c}$$

$$\begin{aligned}\Pr(R_{1,n} \cap A_{1,n}) = {}& u_{1111}^{(n)} + \alpha u_{1121}^{(n)} + (1 - \alpha)\, u_{1212}^{(n)} \\ & + \tfrac{1}{2}\alpha[u_{1122}^{(n)} + u_{1221}^{(n)}] \\ & + (1 - \tfrac{1}{2}\alpha)(u_{1112}^{(n)} + u_{1211}^{(n)}),\end{aligned} \tag{93d}$$

$$\begin{aligned}\Pr(R_{2,n} \cap A_{1,n}) = {}& u_{2111}^{(n)} + \alpha u_{2212}^{(n)} + (1 - \alpha)u_{2121}^{(n)} \\ & + \tfrac{1}{2}(1 - \alpha)[u_{2122}^{(n)} + u_{2221}^{(n)}] \\ & + [1 - \tfrac{1}{2}(1 - \alpha)][u_{2112}^{(n)} + u_{2211}^{(n)}].\end{aligned} \tag{93e}$$

The first equation gives the probability of an R_1-response. The second and third equations give the probability of an A_1-response on T_1- and T_2-trials, respectively. Finally, the last two equations present the probability of the joint occurrence of each observing response with an A_1-response.

In the experiment reported by Atkinson (1961) six groups with 40 subjects in each group were run. For all groups $\pi_1 = 0.9$ and $\beta = 0.5$. The groups differed with respect to the value of α and π_2. For Groups I to III the value of $\alpha = 1$; and for Groups IV to VI $\alpha = 0.75$. For

Groups I and IV, $\pi_2 = 0.9$; for II and V, $\pi_2 = 0.5$; and for Groups III and VI, $\pi_2 = 0.1$. The design can be described by the following array:

		π_2		
		0.9	0.5	0.1
α	1.0	I	II	III
	0.75	IV	V	VI

Given these values of π_1, π_2, α, and β, the 16-state Markov chain is irreducible and aperiodic. Thus $\lim u_{ijkl}^{(n)} = u_{ijkl}$ exists and can be obtained by solving the appropriate set of 16 linear equations (see Eq. 16).

Table 9 Predicted and Observed Asymptotic Response Probabilities in Observing Response Experiment

	Group I			Group II			Group III		
	Pred.	Obs.	SD	Pred.	Obs.	SD	Pred.	Obs.	SD
$\Pr(A_1 \mid T_1)$	0.90	0.94	0.014	0.81	0.85	0.164	0.79	0.79	0.158
$\Pr(A_1 \mid T_2)$	0.90	0.94	0.014	0.59	0.61	0.134	0.21	0.23	0.182
$\Pr(R_1)$	0.50	0.45	0.279	0.55	0.59	0.279	0.73	0.70	0.285
$\Pr(R_1 \cap A_1)$	0.45	0.43	0.266	0.39	0.42	0.226	0.37	0.36	0.164
$\Pr(R_2 \cap A_1)$	0.45	0.47	0.293	0.31	0.31	0.232	0.13	0.16	0.161
	Group IV			**Group V**			**Group VI**		
	Pred.	Obs.	SD	Pred.	Obs.	SD	Pred.	Obs.	SD
$\Pr(A_1 \mid T_1)$	0.90	0.93	0.063	0.80	0.82	0.114	0.73	0.73	0.138
$\Pr(A_1 \mid T_2)$	0.90	0.95	0.014	0.60	0.68	0.114	0.27	0.25	0.138
$\Pr(R_1)$	0.49	0.50	0.257	0.52	0.53	0.305	0.63	0.72	0.263
$\Pr(R_1 \cap A_1)$	0.44	0.47	0.241	0.35	0.38	0.219	0.32	0.36	0.138
$\Pr(R_2 \cap A_1)$	0.46	0.47	0.247	0.34	0.36	0.272	0.19	0.13	0.168

The values predicted by the model are given in Table 9 for the case in which $c = c'$. Values for the u_{ijkl}'s were computed and then combined by Eq. 93 to predict the response probabilities. By presenting a single value for each theoretical quantity in the table we imply that these predictions are independent of c and c'. Actually, this is not always the case. However, for the schedules employed in this experiment the dependency of these asymptotic predictions on c and c' is virtually negligible. For $c = c'$, ranging over the interval from 0.0001 to 1.0, the predicted values given in

Table 9 are affected in only the third or fourth decimal place; it is for this reason that we present theoretical values to only two decimal places.

In view of these comments it should be clear that the predictions in Table 9 are based solely on the experimental parameter values. Consequently, differences between subjects (that may be represented by intersubject variability in c and c') do not substantially affect these predictions.

In the Atkinson study 400 trials were run and the response proportions appear to have reached a fairly stable level over the second half of the experiment. Consequently, the proportions computed over the final block of 160 trials were used as estimates of asymptotic quantities. Table 9 presents the mean and standard deviation of the 40 observed proportions obtained under each experimental condition.

Despite the fact that these gross asymptotic predictions hold up quite well, it is obvious that some of the predictions from the model will not be confirmed. The difficulty with the one-element assumption is that the fundamental theory laid down by the axioms of Sec. 2 is completely deterministic in many respects. For example, when $N' = 1$, we have

$$\Pr(R_{1,n+1} \mid O_{1,n}A_{1,n}R_{1,n}) = 1;$$

namely, if an R_1 occurs on trial n and is reinforced (i.e., followed by an A_1O_1-event), then R_1 will recur with probability 1 on trial $n + 1$. This prediction is, of course, a consequence of the assumption that we have but one element in set S_R which necessarily is sampled on every trial. If we assume more than one element, the deterministic features of the model no longer hold, and such sequential statistics become functions of c, c', N, and N'. Unfortunately, for elaborate experimental procedures of the sort described in this section the multi-element case leads to complicated mathematical processes for which it is extremely difficult to carry out computations. Thus the generality of the multi-element assumption may often be offset by the difficulty involved in making predictions.

Naturally, it is usually preferable to choose from the available models the one that best fits the data, but in the present state of psychological knowledge no single model is clearly superior to all others in every facet of analysis. The one-element assumption, despite some of its erroneous features, may prove to be a valuable instrument for the rapid exploration of a wide variety of complex phenomena. For most of the cases we have examined the predicted mean response probabilities are usually independent of (or only slightly dependent on) the number of elements assumed. Thus the one-element assumption may be viewed as a simple device for computing the grosser predictions of the general theory.

For exploratory work in complex situations, then, we recommend using the one-element model because of the greater difficulty of computations for the multi-element models. In advocating this approach, we are taking a methodological position with which some scientists do not agree. Our position is in contrast to one that asserts that a model should be discarded once it is clear that certain of its predictions are in error. We do not take it to be the principal goal (or even, in many cases, an important goal) of theory construction to provide models for particular experimental situations. The assumptions of stimulus sampling theory are intended to describe processes or relationships that are common to a wide variety of learning situations but with no implication that behavior in these situations is a function solely of the variables represented in the theory. As we have attempted to illustrate by means of numerous examples, the formulation of a model within this framework for a particular experiment is a matter of selecting the relevant assumptions, or axioms, of the general theory and interpreting them in terms of the conditions of the experiment. How much of the variance in a set of data can be accounted for by a model depends jointly on the adequacy of the theoretical assumptions and on the extent to which it has been possible to realize experimentally the boundary conditions envisaged in the theory, thereby minimizing the effects of variables not represented. In our view a model, in application to a given experiment, is not to be classified as "correct" or "incorrect"; rather, the degree to which it accounts for the data may provide evidence tending either to support or to cast doubt on the theory from which it was derived.

References

Atkinson, R. C. A stochastic model for rote serial learning. *Psychometrika*, 1957, **22,** 87–96.

Atkinson, R. C. A Markov model for discrimination learning. *Psychometrika*, 1958, **23,** 308–322.

Atkinson, R. C. A theory of stimulus discrimination learning. In K. J. Arrow, S. Karlin, & P. Suppes (Eds.), *Mathematical methods in the social sciences.* Stanford: Stanford Univer. Press, 1960. Pp. 221–241.

Atkinson, R. C. The observing response in discrimination learning. *J. exp. Psychol.*, 1961, **62,** 253–262.

Atkinson, R. C. Choice behavior and monetary payoffs. In J. Criswell, H. Solomon, & P. Suppes (Eds.), *Mathematical methods in small group processes.* Stanford: Stanford Univer. Press, 1962. Pp. 23–34.

Atkinson, R. C. Mathematical models in research on perception and learning. In M. Marx (Ed.), *Psychological Theory.* (2nd ed.) New York: Macmillan, 1963, in press. (a)

Atkinson, R. C. A variable sensitivity theory of signal detection. *Psychol. Rev.*, 1963, **70,** 91–106. (b)

Atkinson, R. C., & Suppes, P. An analysis of two-person game situations in terms of statistical learning theory. *J. exp. Psychol.*, 1958, **55,** 369–378.

Billingsley, P. *Statistical inference for Markov processes.* Chicago: Univer. of Chicago Press, 1961.

Bower, G. H. Choice-point behavior. In R. R. Bush & W. K. Estes (Eds.), *Studies in mathematical learning theory.* Stanford: Stanford Univer. Press, 1959. Pp. 109–124.

Bower, G. H. Application of a model to paired-associate learning. *Psychometrika,* 1961, **26,** 255–280.

Bower, G. H. A model for response and training variables in paired-associate learning. *Psychol. Rev.,* 1962, **69,** 34–53.

Burke, C. J. Applications of a linear model to two-person interactions. In R. R. Bush & W. K. Estes (Eds.), *Studies in mathematical learning theory.* Stanford: Stanford Univer. Press, 1959. Pp. 180–203.

Burke, C. J. Some two-person interactions. In K. J. Arrow, S. Karlin, & P. Suppes (Eds.), *Mathematical methods in the social sciences.* Stanford: Stanford Univer. Press, 1960. Pp. 242–253.

Burke, C. J., & Estes, W. K. A component model for stimulus variables in discrimination learning. *Psychometrika,* 1957, **22,** 133–145.

Bush, R. R. A survey of mathematical learning theory. In R. D. Luce (Ed.), *Developments in mathematical psychology.* Glencoe, Illinois: The Free Press, 1960. Pp. 123–165.

Bush, R. R., & Estes, W. K. (Eds.), *Studies in mathematical learning theory.* Stanford: Stanford Univer. Press, 1959.

Bush, R. R., & Mosteller, F. A mathematical model for simple learning. *Psychol. Rev.,* 1951, **58,** 313–323. (a)

Bush, R. R., & Mosteller, F. A model for stimulus generalization and discrimination. *Psychol. Rev.,* 1951, **58,** 413–423. (b)

Bush, R. R., & Mosteller, F. *Stochastic models for learning.* New York: Wiley, 1955.

Bush, R. R., & Sternberg, S. A single-operator model. In R. R. Bush & W. K. Estes (Eds.), *Studies in mathematical learning theory.* Stanford: Stanford Univer. Press, 1959. Pp. 204–214.

Carterette, Teresa S. An application of stimulus sampling theory to summated generalization. *J. exp. Psychol.,* 1961, **62,** 448–455.

Crothers, E. J. *All-or-none paired associate learning with unit and compound responses.* Unpublished doctoral dissertation, Indiana University, 1961.

Detambel, M. H. A test of a model for multiple-choice behavior. *J. exp. Psychol.,* 1955, **49,** 97–104.

Estes, W. K. Toward a statistical theory of learning. *Psychol. Rev.,* 1950, **57,** 94–107.

Estes, W. K. Statistical theory of spontaneous recovery and regression. *Psychol. Rev.,* 1955, **62,** 145–154. (a)

Estes, W. K. Statistical theory of distributional phenomena in learning. *Psychol. Rev.,* 1955, **62,** 369–377. (b)

Estes, W. K. Of models and men. *Amer. Psychol.,* 1957, **12,** 609–617. (a)

Estes, W. K. Theory of learning with constant, variable, or contingent probabilities of reinforcement. *Psychometrika,* 1957, **22,** 113–132. (b)

Estes, W. K. Stimulus-response theory of drive. In M. R. Jones (Ed.), *Nebraska symposium on motivation. Vol.* 6. Lincoln, Nebraska: Univer. Nebraska Press, 1958.

Estes, W. K. The statistical approach to learning theory. In S. Koch (Ed.), *Psychology: a study of a science. Vol.* 2. New York: McGraw-Hill, 1959. Pp. 380–491. (a)

Estes, W. K. Component and pattern models with Markovian interpretations. In R. R. Bush & W. K. Estes (Eds.), *Studies in mathematical learning theory.* Stanford: Stanford Univer. Press, 1959. Pp. 9–52. (b)

Estes, W. K. Learning theory and the new mental chemistry. *Psychol. Rev.*, 1960, **67,** 207–223. (a)

Estes, W. K. A random-walk model for choice behavior. In K. J. Arrow, S. Karlin, & P. Suppes (Eds.), *Mathematical methods in the social sciences.* Stanford: Stanford Univer. Press, 1960. Pp. 265–276. (b)

Estes, W. K. Growth and function of mathematical models for learning. In *Current trends in psychological theory.* Pittsburgh: Univer. of Pittsburgh Press, 1961. Pp. 134–151. (a)

Estes, W. K. New developments in statistical behavior theory: differential tests of axioms for associative learning. *Psychometrika*, 1961, **26,** 73–84. (b)

Estes, W. K. Learning theory. *Ann. Rev. Psychol.*, 1962, **13,** 107–144.

Estes, W. K., & Burke, C. J. A theory of stimulus variability in learning. *Psychol. Rev.*, 1953, **60,** 276–286.

Estes, W. K., Burke, C. J., Atkinson, R. C., & Frankmann, Judith P. Probabilistic discrimination learning. *J. exp. Psychol.*, 1957, **54,** 233–239.

Estes, W. K., & Hopkins, B. L. Acquisition and transfer in pattern -vs.- component discrimination learning. *J. exp. Psychol.*, 1961, **61,** 322–328.

Estes, W. K., Hopkins, B. L., & Crothers, E. J. All-or-none and conservation effects in the learning and retention of paired associates. *J. exp. Psychol.*, 1960, **60,** 329–339.

Estes, W. K., & Straughan, J. H. Analysis of a verbal conditioning situation in terms of statistical learning theory. *J. exp. Psychol.*, 1954, **47,** 225–234.

Estes, W. K., & Suppes, P. Foundations of linear models. In R. R. Bush & W. K. Estes (Eds.), *Studies in mathematical learning theory.* Stanford: Stanford Univer. Press, 1959. Pp. 137–179. (a)

Estes, W. K., & Suppes, P. *Foundations of statistical learning theory, II. The stimulus sampling model for simple learning.* Tech. Rept. No. 26, Psychology Series, Institute for Mathematical Studies in the Social Sciences, Stanford Univer., 1959. (b)

Feller, W. *An introduction to probability theory and its applications.* (2nd ed.) New York: Wiley, 1957.

Friedman, M. P., Burke, C. J., Cole, M., Estes, W. K., & Millward, R. B. *Extended training in a noncontingent two-choice situation with shifting reinforcement probabilities.* Paper given at the First Meetings of the Psychonomic Society, Chicago, Illinois, 1960.

Gardner, R. A. Probability-learning with two and three choices. *Amer. J. Psychol.*, 1957, **70,** 174–185.

Guttman, N., & Kalish, H. I. Discriminability and stimulus generalization. *J. exp. Psychol.*, 1956, **51,** 79–88.

Goldberg, S. *Introduction to difference equations.* New York: Wiley, 1958.

Hull, C. L. *Principles of behavior: an introduction to behavior theory.* New York: Appleton-Century-Crofts, 1943.

Jarvik, M. E. Probability learning and a negative recency effect in the serial anticipation of alternating symbols. *J. exp. Psychol.*, 1951, **41,** 291–297.

Jordan, C. *Calculus of finite differences.* New York: Chelsea, 1950.

Kemeny, J. G., & Snell, J. L. Markov processes in learning theory. *Psychometrika*, 1957, **22,** 221–230.

Kemeny, J. G., & Snell, J. L. *Finite Markov chains.* Princeton, N. J.: Van Nostrand, 1959.

Kemeny, J. G., Snell, J. L., & Thompson, G. L. *Introduction to finite mathematics.* New York: Prentice Hall, 1957.

Kinchla, R. A. *Learned factors in visual discrimination.* Unpublished doctoral dissertation, Univer. of California, Los Angeles, 1962.

Lamperti, J., & Suppes, P. Chains of infinite order and their applications to learning theory. *Pacific J. Math.*, 1959, **9**, 739–754.

Luce, R. D. *Individual choice behavior: a theoretical analysis*. New York: Wiley, 1959.

Luce, R. D. A threshold theory for simple detection experiments. *Psychol. Rev.*, 1963, **70**, 61–79.

Luce, R. D., & Raiffa, H. *Games and decisions*. New York: Wiley, 1957.

Nicks, D. C. Prediction of sequential two-choice decisions from event runs. *J. exp. Psychol.*, 1959, **57**, 105–114.

Peterson, L. R., Saltzman, Dorothy, Hillner, K., & Land, Vera. Recency and frequency in paired-associate learning. *J. exp. Psychol.*, 1962, **63**, 396–403.

Popper, Juliet. Mediated generalization. In R. R. Bush & W. K. Estes, (Eds.), *Studies in mathematical learning theory*. Stanford: Stanford Univer. Press, 1959. Pp. 94–108.

Popper, Juliet, & Atkinson, R. C. Discrimination learning in a verbal conditioning situation. *J. exp. Psychol.*, 1958, **56**, 21–26.

Restle, F. A theory of discrimination learning. *Psychol. Rev.*, 1955, **62**, 11–19.

Restle, F. *Psychology of judgment and choice*. New York: Wiley, 1961.

Solomon, R. L., & Wynne, L. C. Traumatic avoidance learning: acquisition in normal dogs. *Psychol. Monogr.*, 1953, **67**, No. 4.

Spence, K. W. The nature of discrimination learning in animals. *Psychol. Rev.*, 1936, **43**, 427–449.

Stevens, S. S. On the psychophysical law. *Psychol. Rev.*, 1957, **64**, 153–181.

Suppes, P., & Atkinson, R. C. *Markov learning models for multiperson interactions*. Stanford: Stanford Univer. Press, 1960.

Suppes, P., & Ginsberg, Rose. Application of a stimulus sampling model to children's concept formation of binary numbers with and without an overt correction response. *J. exp. Psychol.*, 1962, **63**, 330–336.

Suppes, P., & Ginsberg, Rose. A fundamental property of all-or-none models. *Psychol. Rev.*, 1963, **70**, 139–161.

Swets, J. A., Tanner, W. P., Jr., & Birdsall, T. G. Decision processes in perception. *Psychol. Rev.*, 1961, **68**, 301–340.

Tanner, W. P., Jr., & Swets, J. A. A decision-making theory of visual detection. *Psychol. Rev.*, 1954, **61**, 401–409.

Theios, J. Simple conditioning as two-stage all-or-none learning. *Psychol. Rev.*, 1963, in press.

Wyckoff, L. B., Jr. The role of observing responses in discrimination behavior. *Psychol. Rev.*, 1952, **59**, 431–442.